Physique mesoscopique des electrons et des photons -
Structures fractales et quasi-periodiques

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Physics-Technion

Aux frontieres de la physique mesoscopique,
Mont Orford Quebec, Canada,
Septembre 2013

Part 3

Quantum mesoscopic physics :
Fractals and quasi-periodic structures

Fractals or the skill of playing with dimensionality

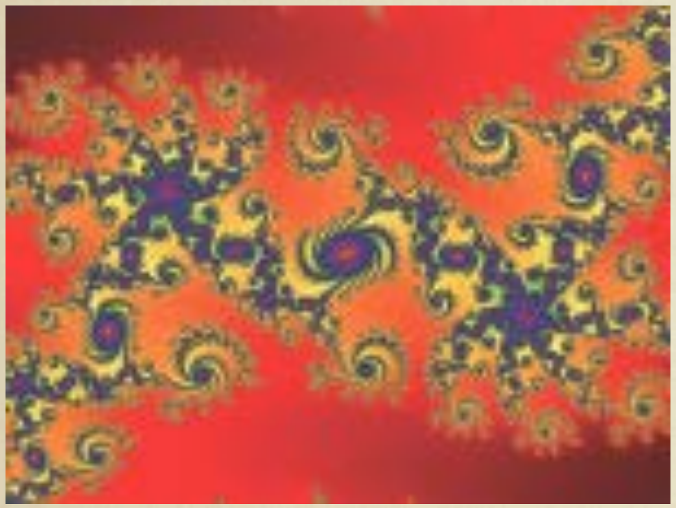
Why studying fractals in physics ?

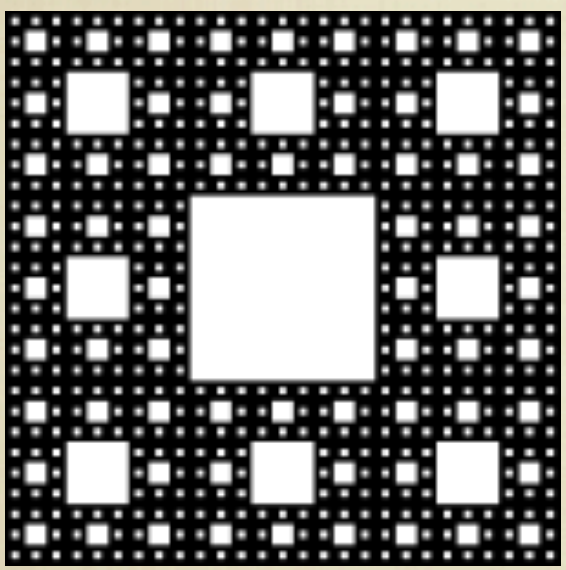
Fractals define a very useful testing ground for dimensionality dependent physical problems since distinct physical properties are characterized by different (usually non integer) dimensions.

Some examples

- Anderson localization phase transition : exists for $d > 2$
- Bose-Einstein condensation ($d \geq 3$)
- Mermin-Wagner theorem (Superfluidity $d \leq 2$)
- Levy flights-Percolation (quantum and classical)
- Recurrence properties of random walks
- Quantum mesoscopic physics
- Quantum and classical phase transitions-Existence of topological defects...

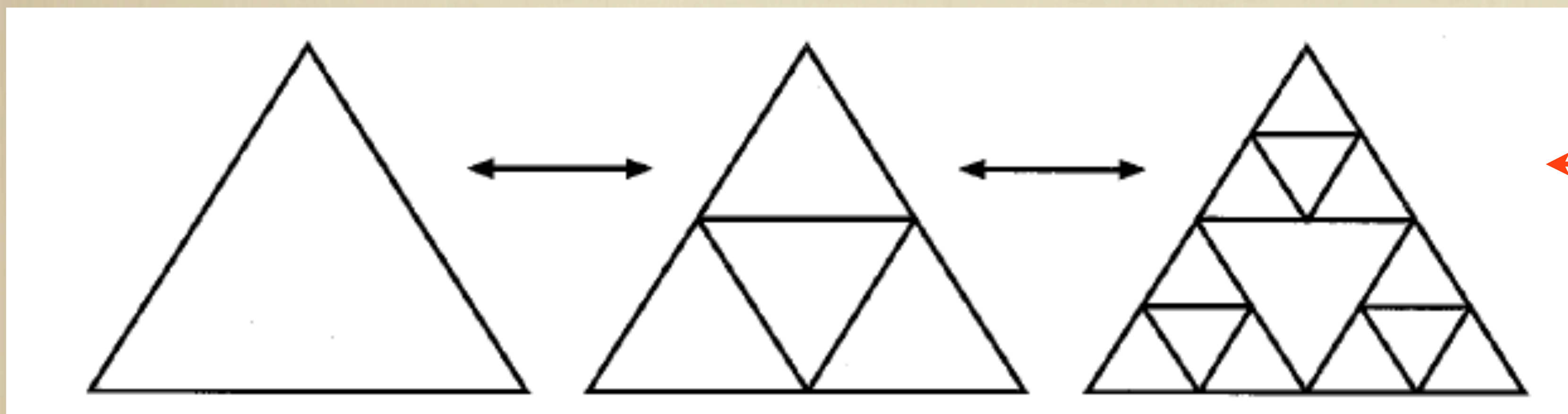
Fractals are also interesting from a practical point of view (in addition to providing nice pictures...)

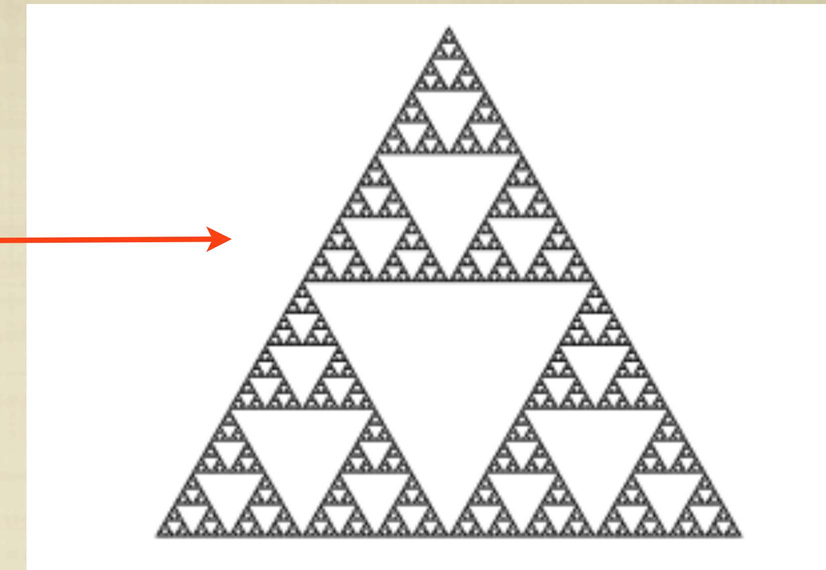




Random lasers : pumping on randomly localized modes (difficult to locate them).

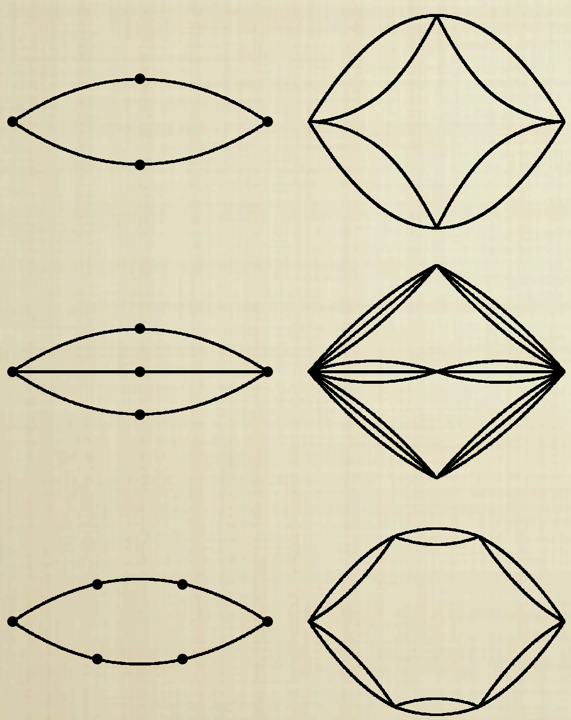
Iterative fractal graph structure

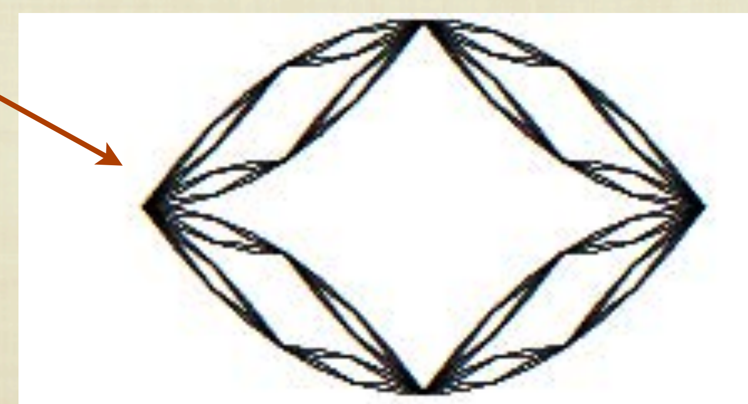


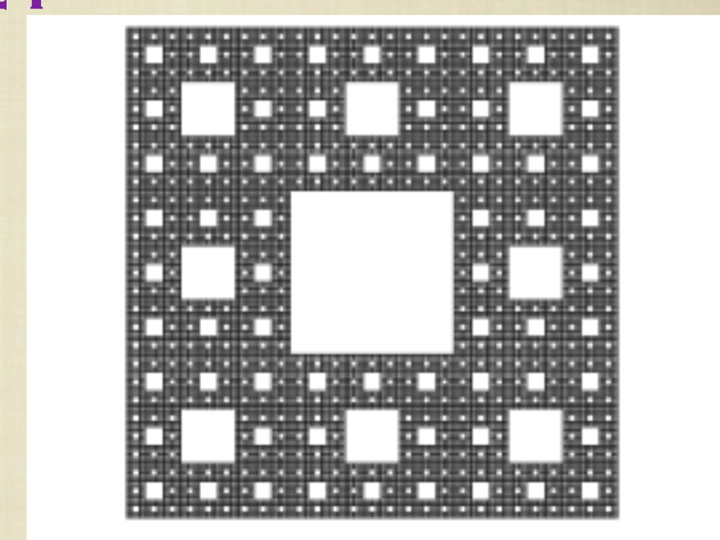


$n=0$ $n=1$ $n=2$

Sierpinski gasket





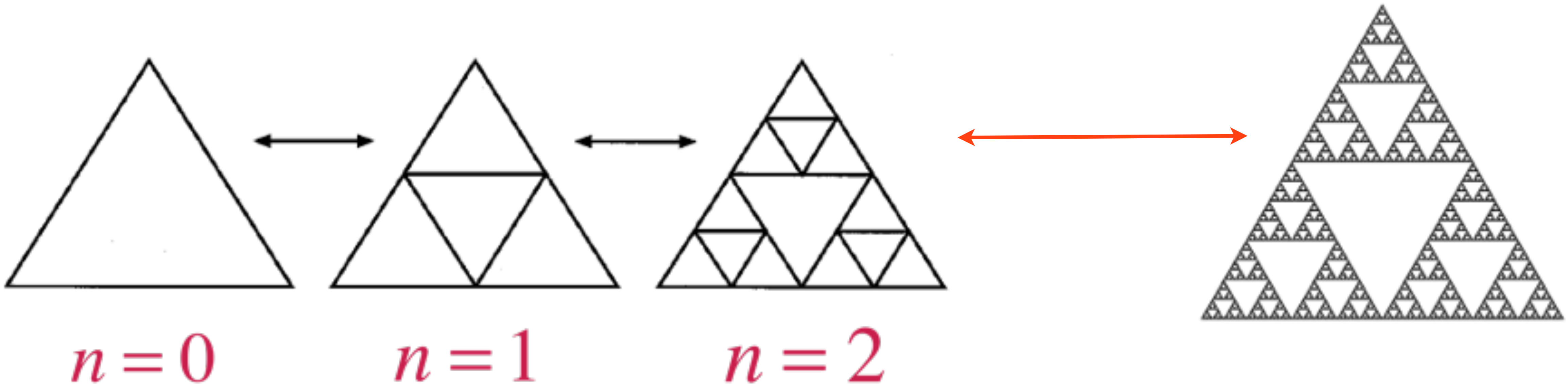


Sierpinski carpet

Diamond fractals

As opposed to Euclidean spaces characterized by translation symmetry, fractals possess a dilatation symmetry of their physical properties, each characterized by a specific fractal dimension.

At each step n of the iteration, a fractal is characterized by its total length L_n and a number of sites N_n.
Scaling of these dimensionless quantities allows to define fractal dimensions.



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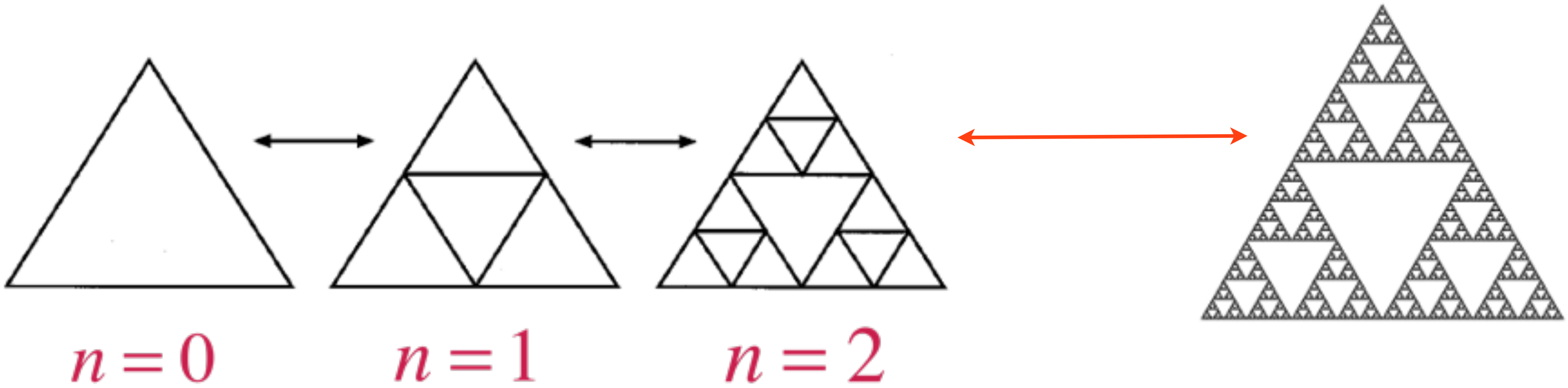

Example: Spatial Hausdorff dimension

$$d_h = \frac{\ln N_n}{\ln L_n}$$

On a Sierpinski gasket $N(2L) = 3N(L)$

so that $d_h = \frac{\ln 3}{\ln 2}$ ~1.585

Classical diffusion

Consider on an Euclidean manifold, the mean square displacement

$$\left\langle r^2(t) \right\rangle = Dt$$

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Another fractal dimension distinct from $d_h = \dfrac{\ln N_n}{\ln L_n}$

Quantum mesoscopic physics on Fractals :

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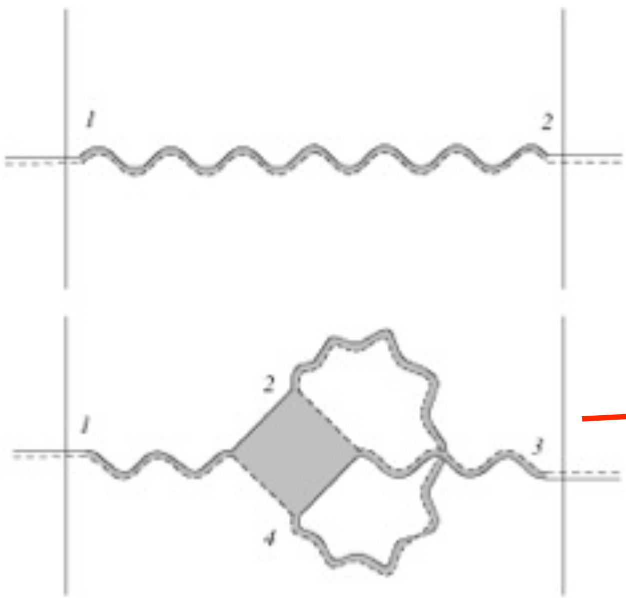
Could repeat all we did before but on a fractal...

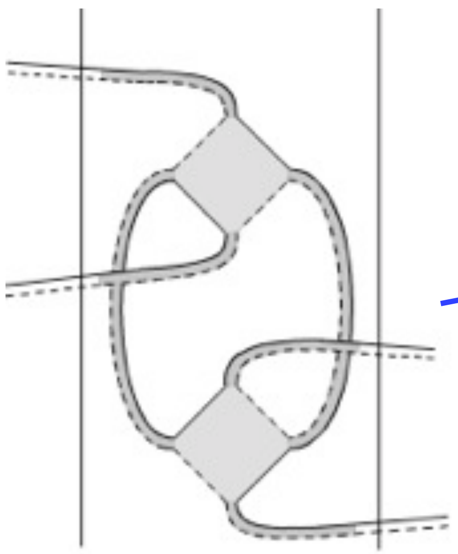
Summary ... and closed loops :



Weak localization corrections to the electrical conductance

$$\frac{\Delta G}{G_{cl}} \propto -\frac{1}{g}\int_0^{\tau_D} Z(t)\frac{dt}{\tau_D}$$

Conductance fluctuations



$$\frac{\overline{\delta G^2}}{G_{cl}^2} \propto \frac{1}{g^2}\int_0^{\tau_D} Z(t)\frac{t\,dt}{\tau_D^2}$$

$$Z(t) = \int dr\, P_{cl}(r,r,t) = \left(\frac{\tau_D}{4\pi t}\right)^{d/2}$$

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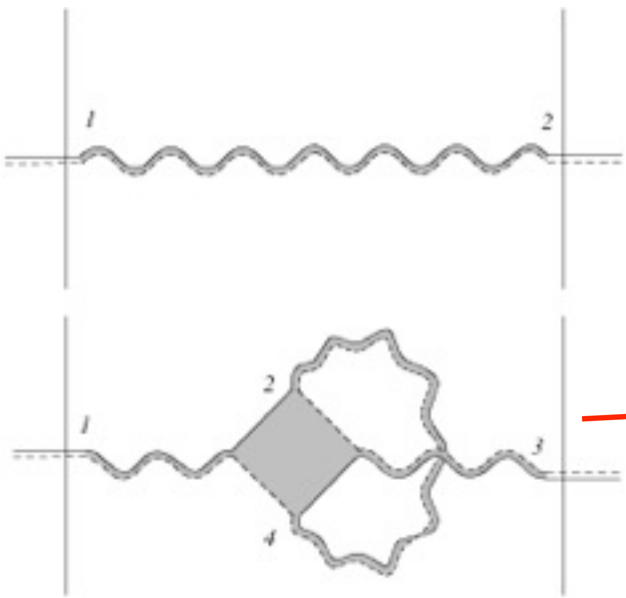

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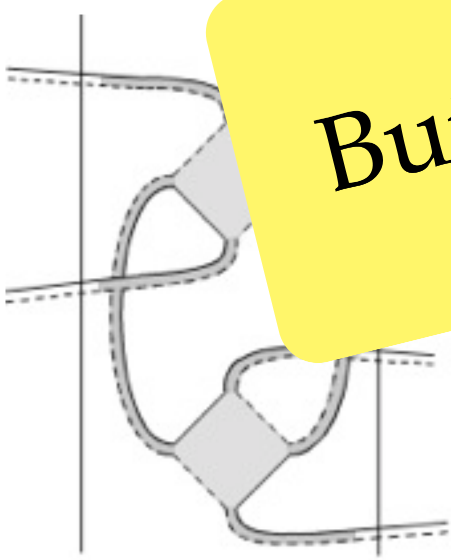
But with Z(t) calculated on a fractal



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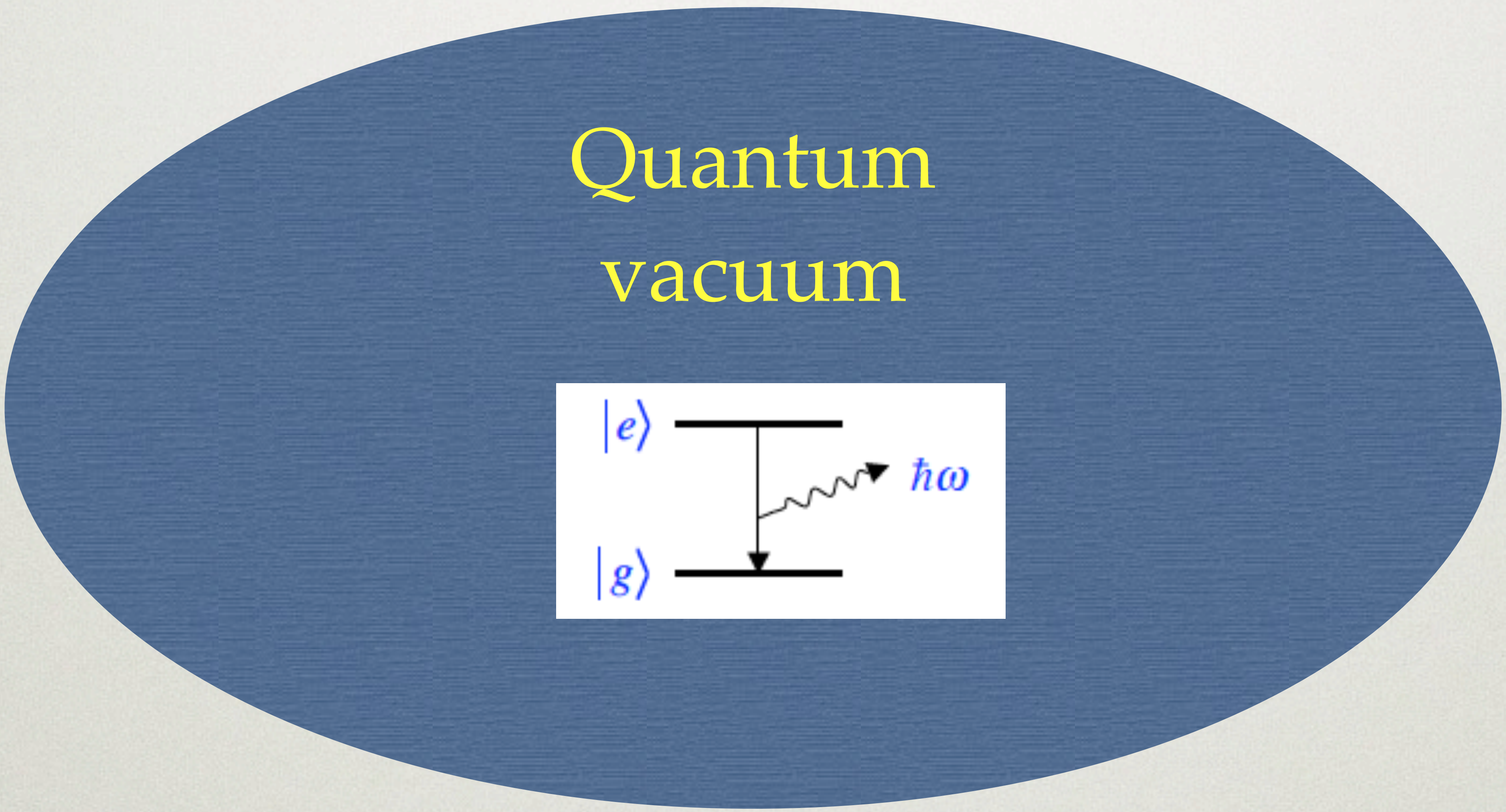
Spontaneous emission - Energy spectra and dynamics on fractals

Spontaneous emission from a fractal QED spectrum



A large variety of problems are conveniently described in terms of spectral classes

(absolutely continuous / singular-continuous / point spectrum):

- Anderson localization
- Quantum and classical wave diffusion
- Random magnetism
- …

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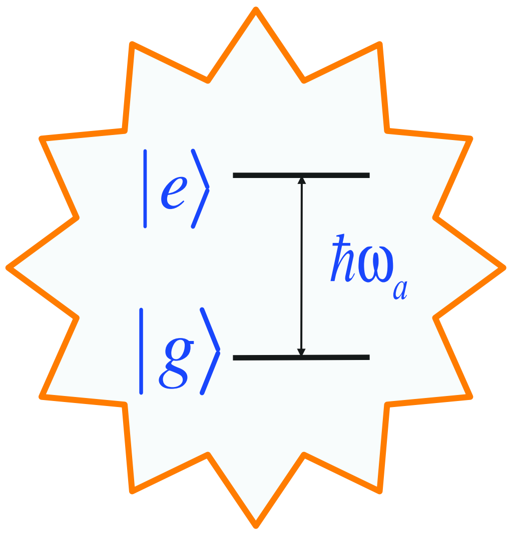
(absolutely continuous / ... trum):

- ...
- ...usion
- ...
- ...

What about a fractal QED vacuum and spontaneous emission ?

Spontaneous emission for different QED vacua



atom

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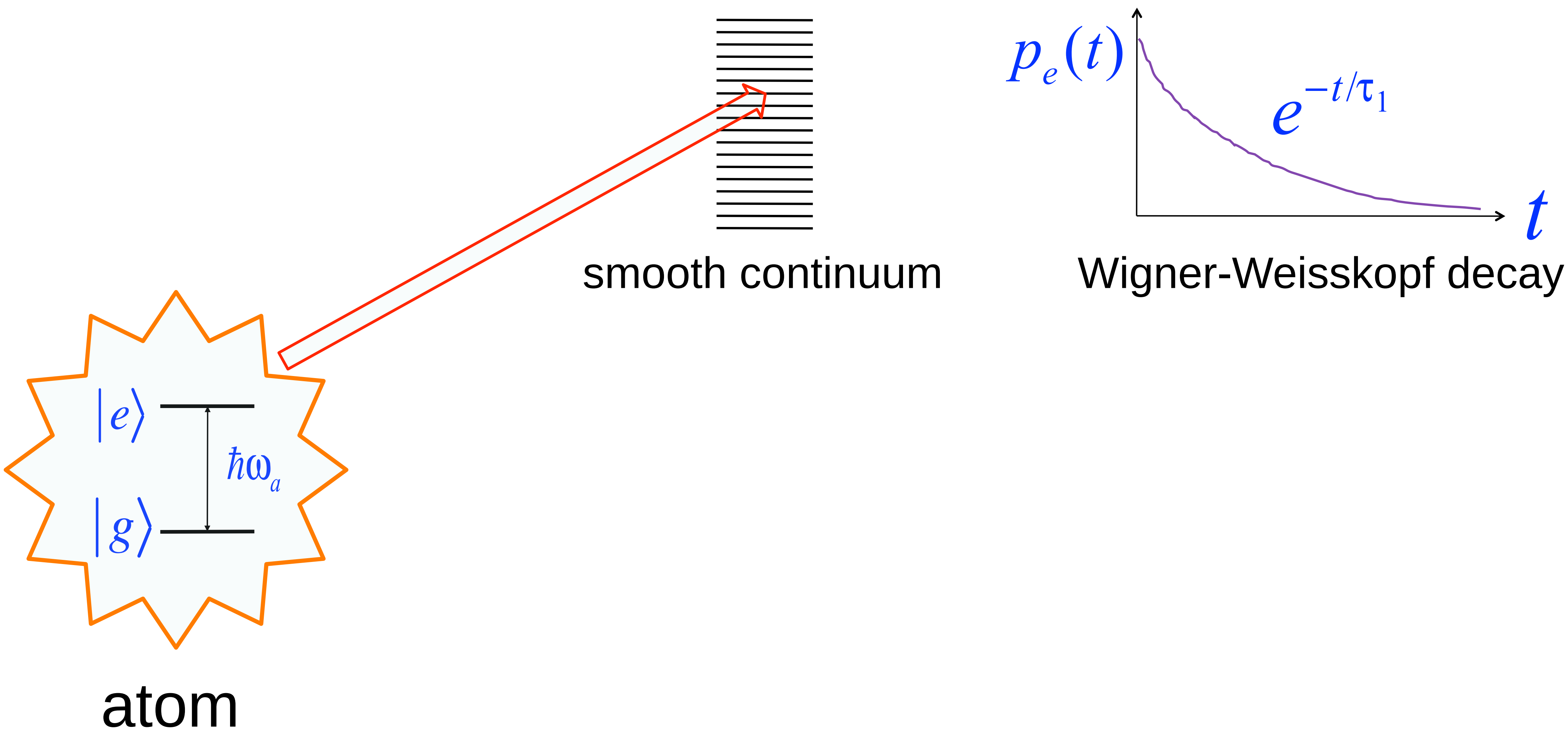

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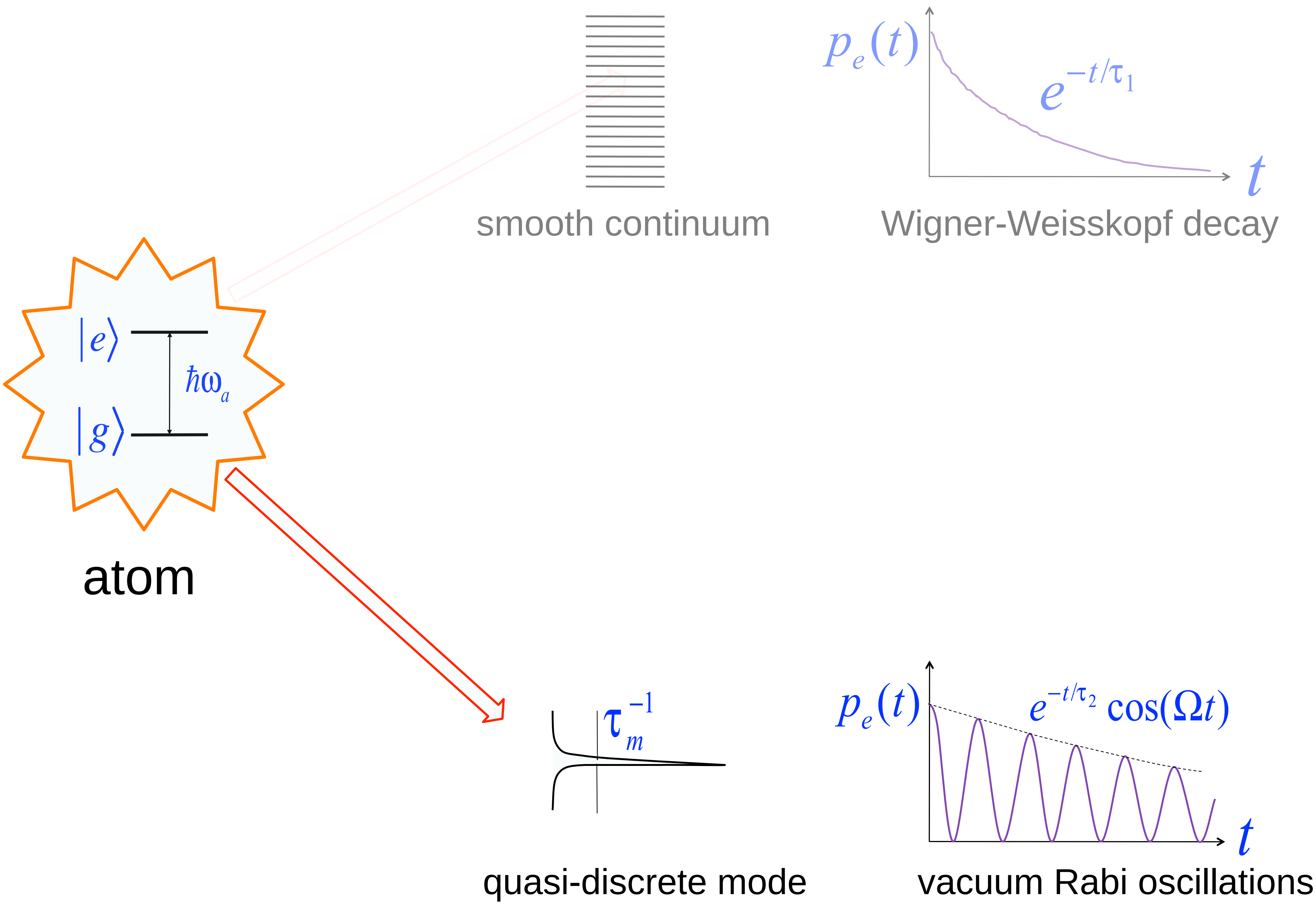

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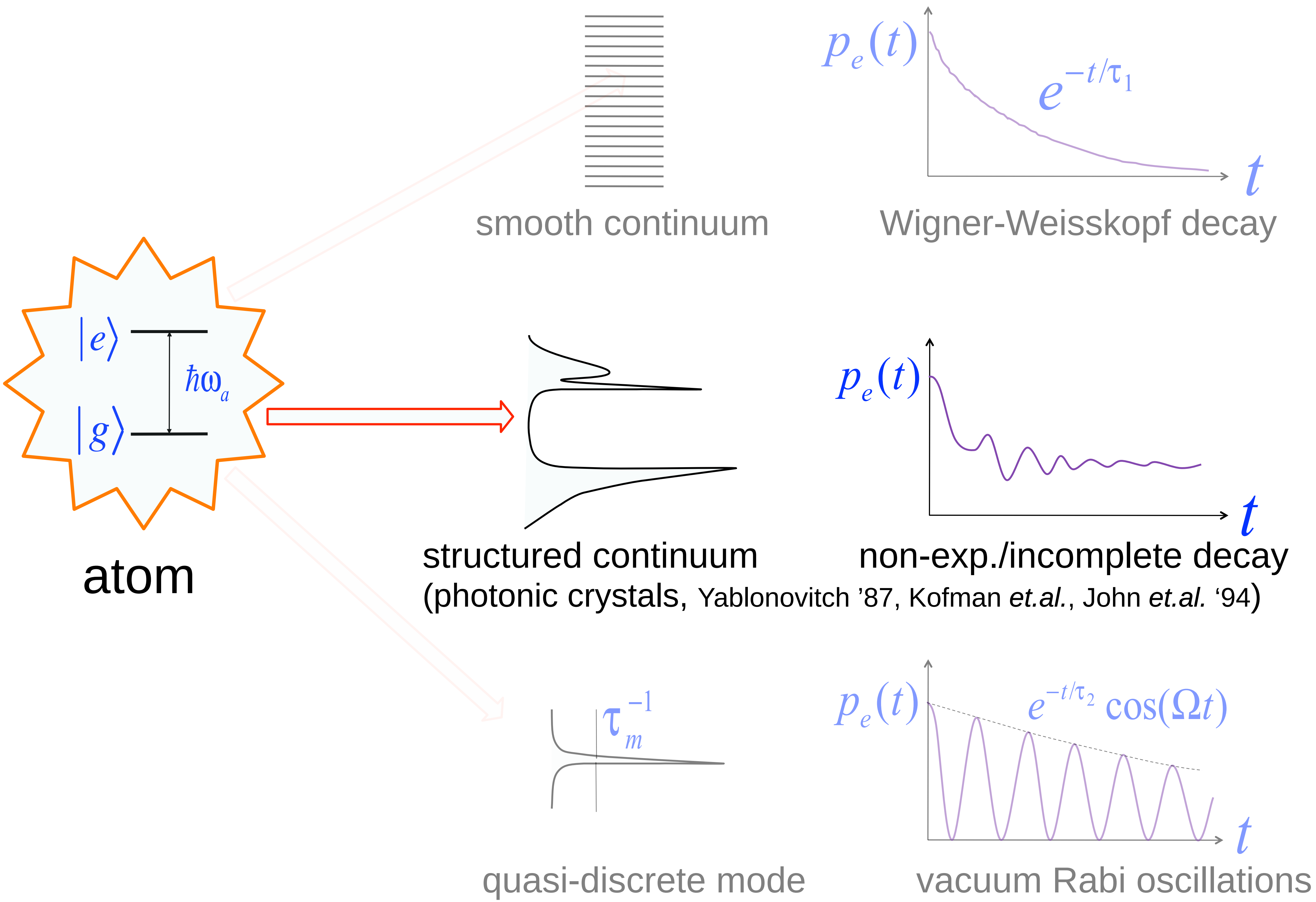

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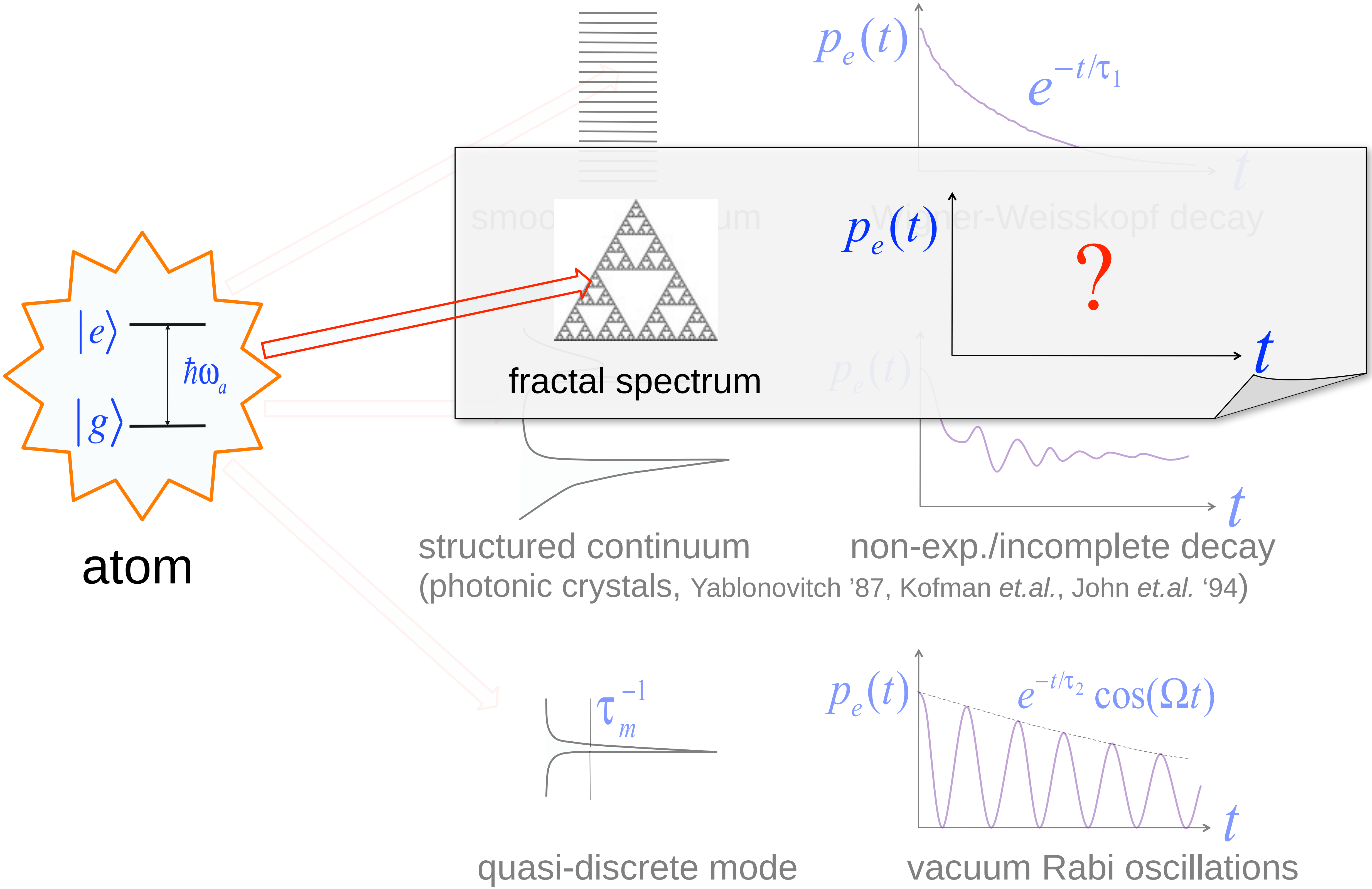
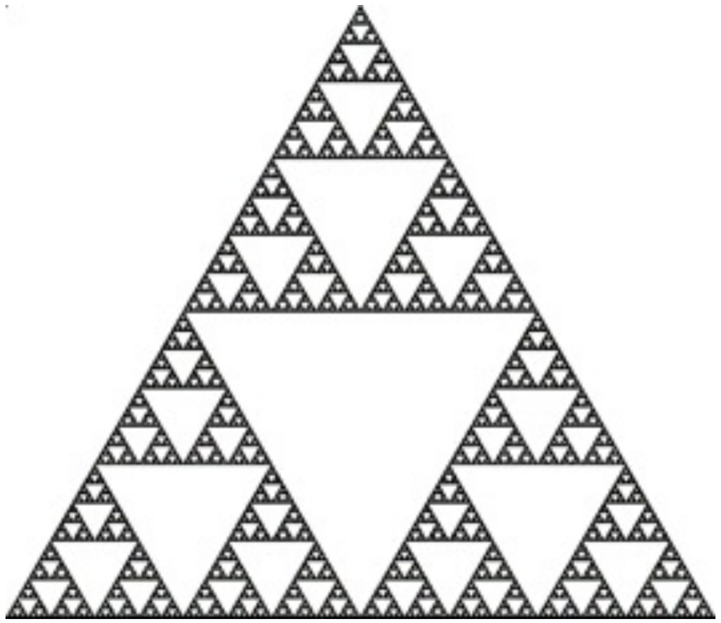


Fractal spectrum : what is it ?

Fractal ↔ Self-similar



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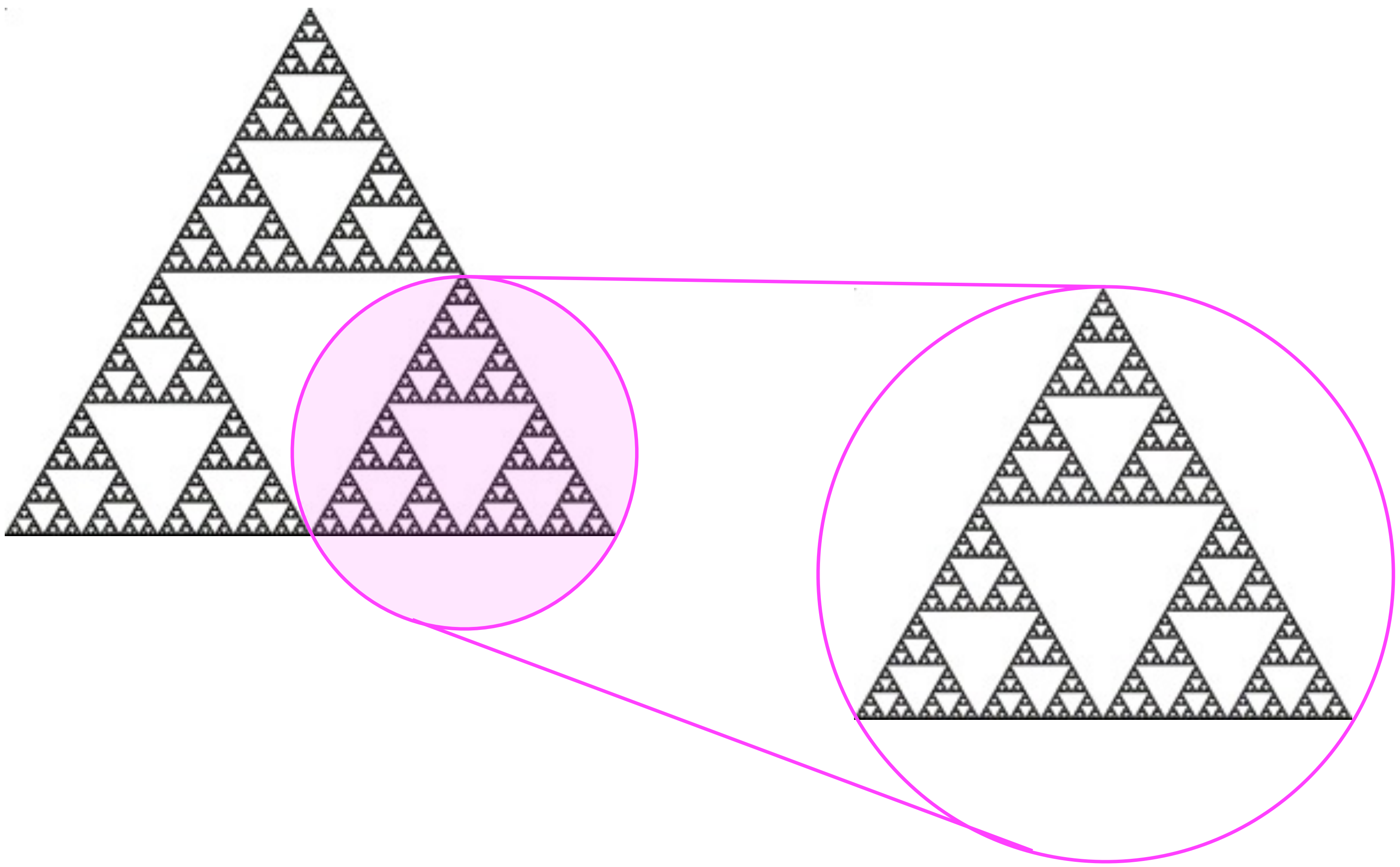

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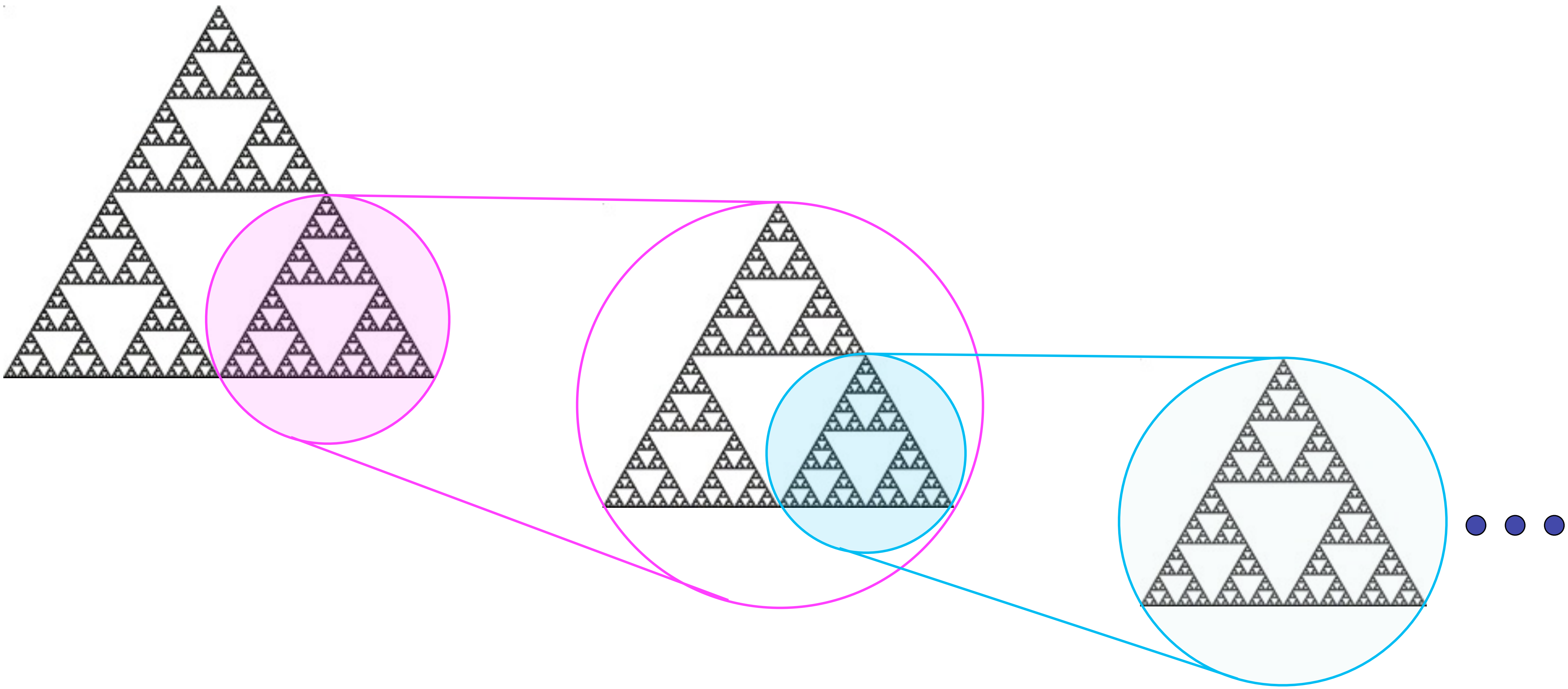
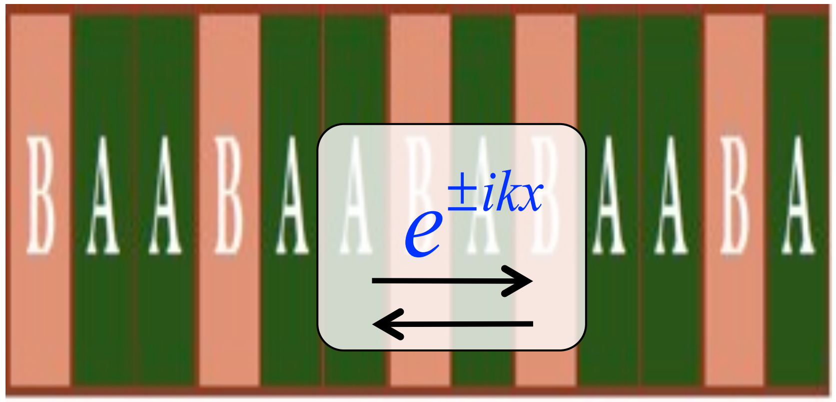


Discrete scaling symmetry

Fractal spectrum - an example

A quasi-periodic stack of dielectric layers of two types (n_A,n_B)



(Kohmoto *et. al.*, '87)

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Fibonacci sequence: $S_{j\geq 2} = \left[S_{j-1} S_{j-2} \right], \quad S_0 = B, \quad S_1 = A$

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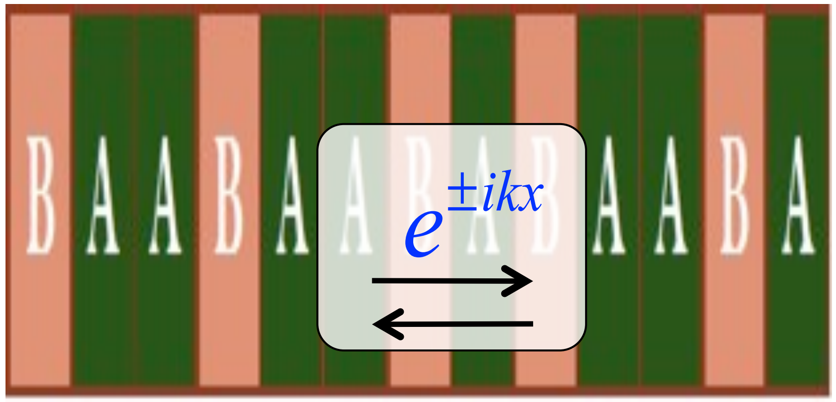

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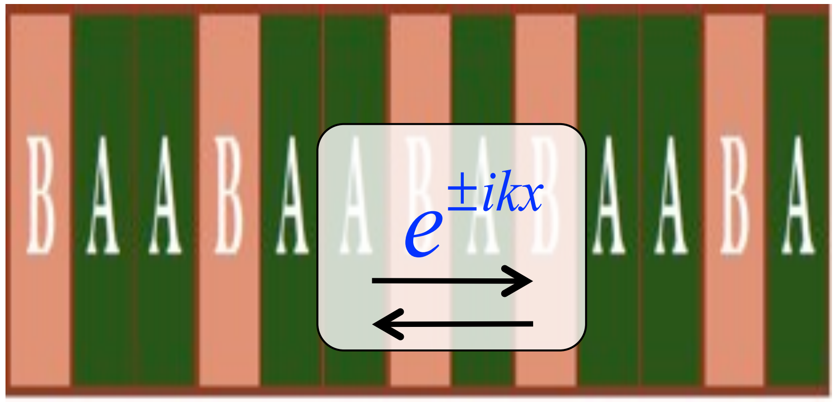

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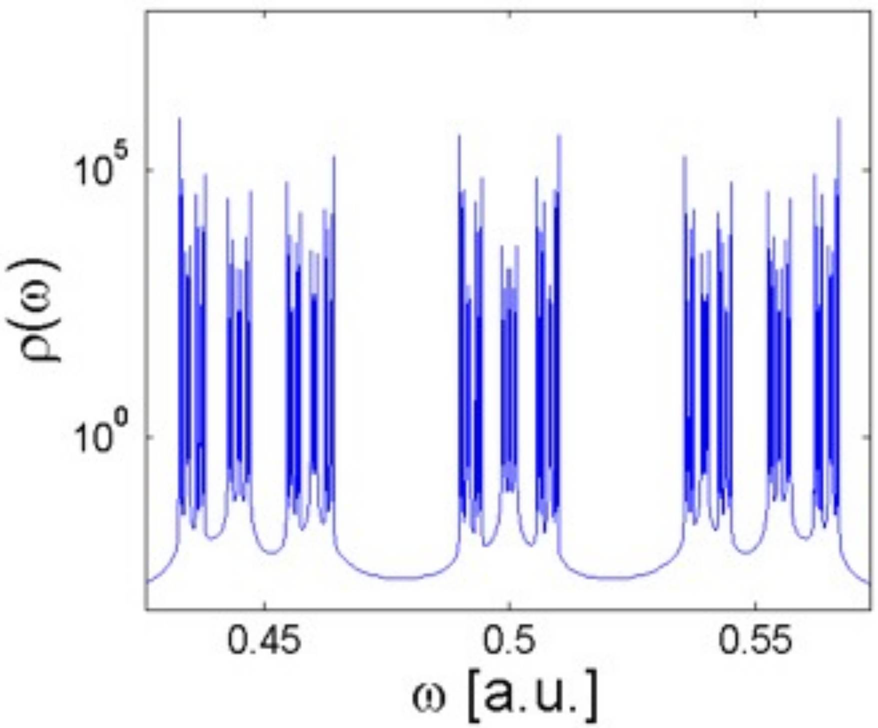
The density of modes $\rho(\omega)$:



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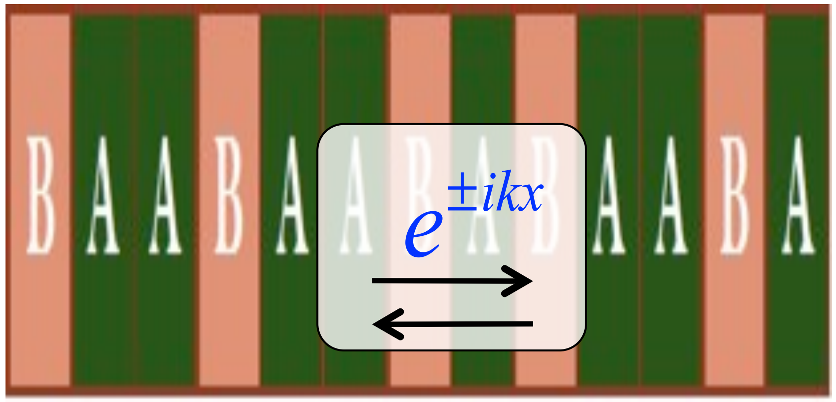

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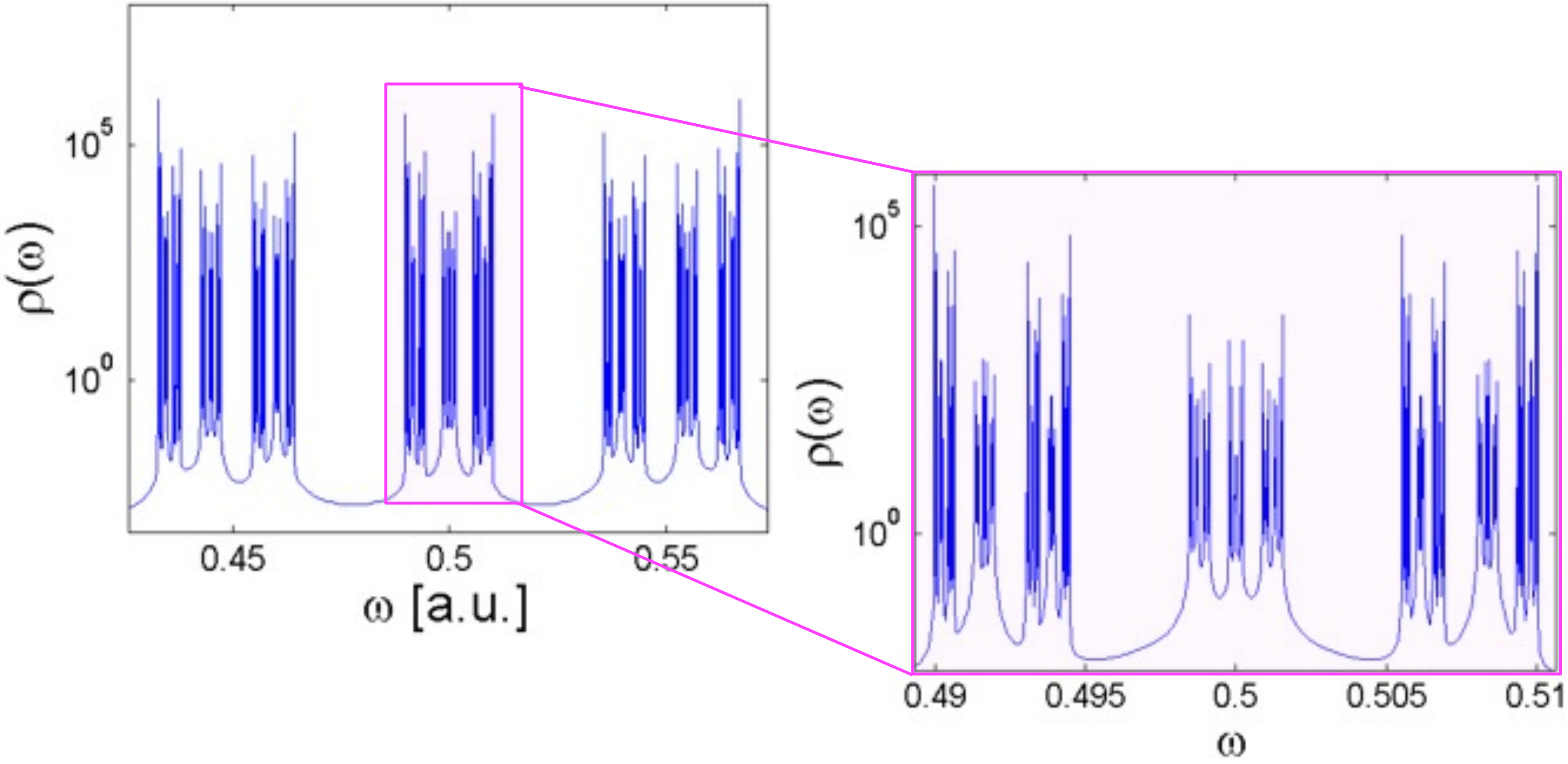

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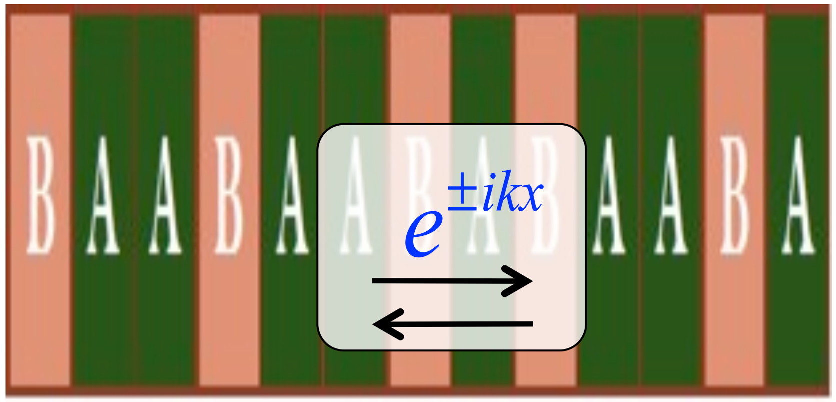

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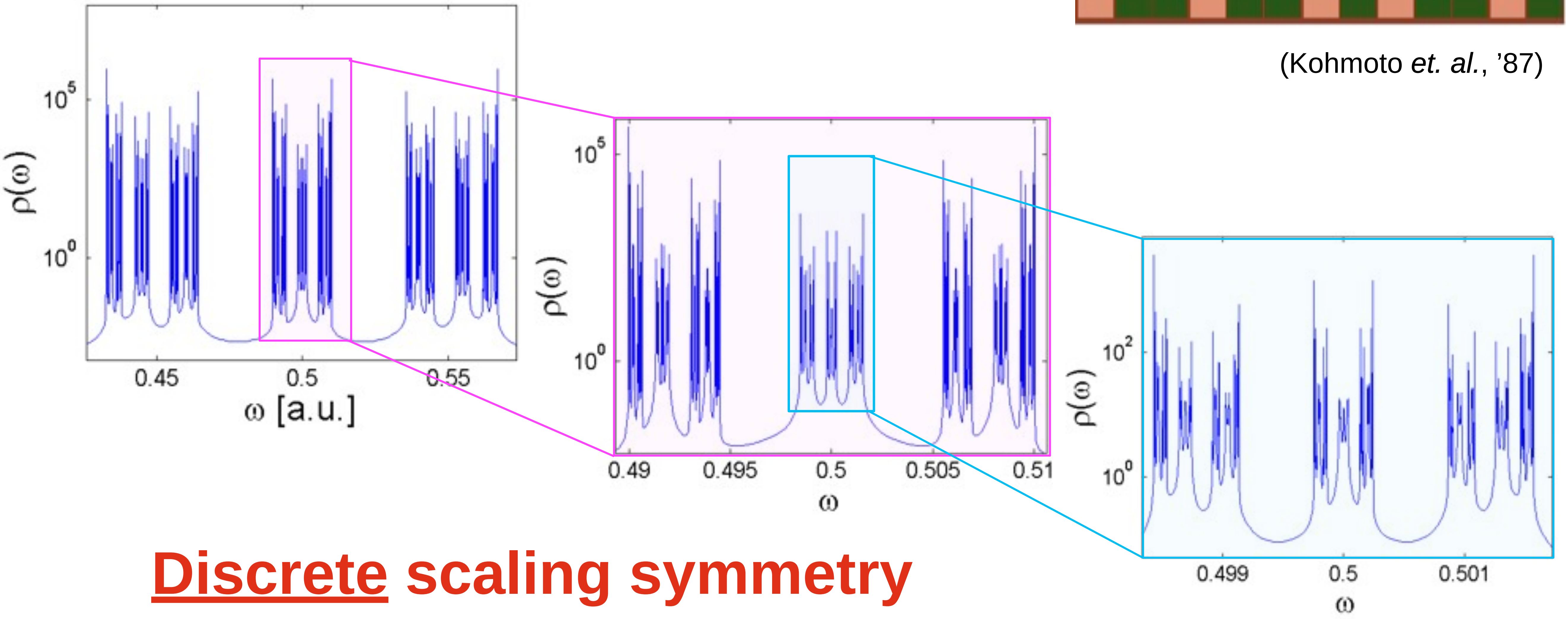
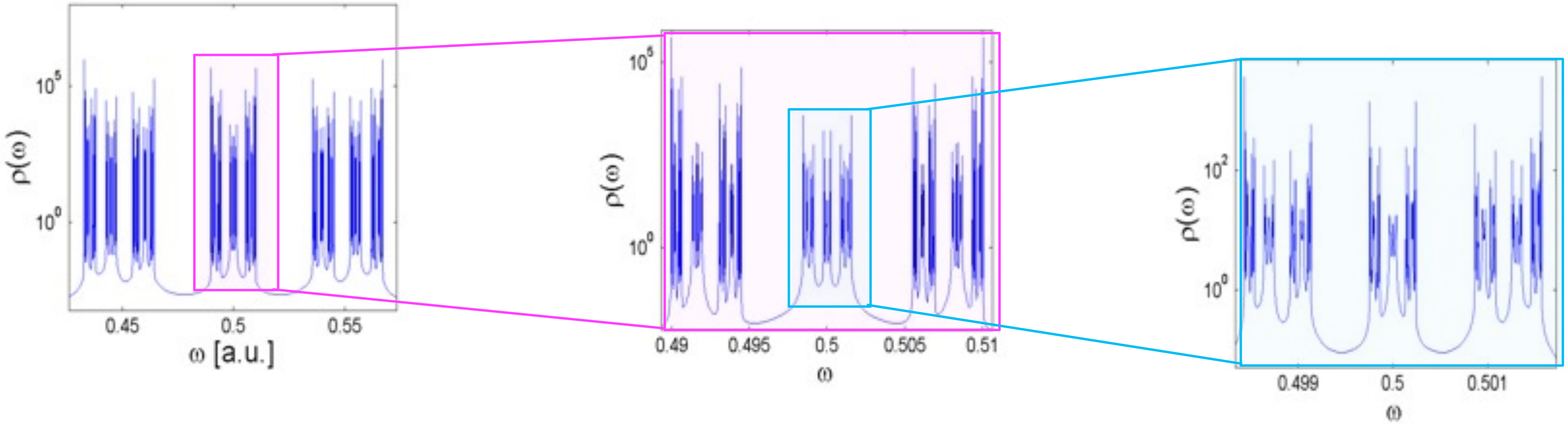


<u>Discrete</u> scaling symmetry

The notion of Fibonacci structure is of broad interest in various fields...

Courtesy of Gerald Dunne for today’s talk
(from Adelaide, Australia)

Discrete scaling symmetry: formal description



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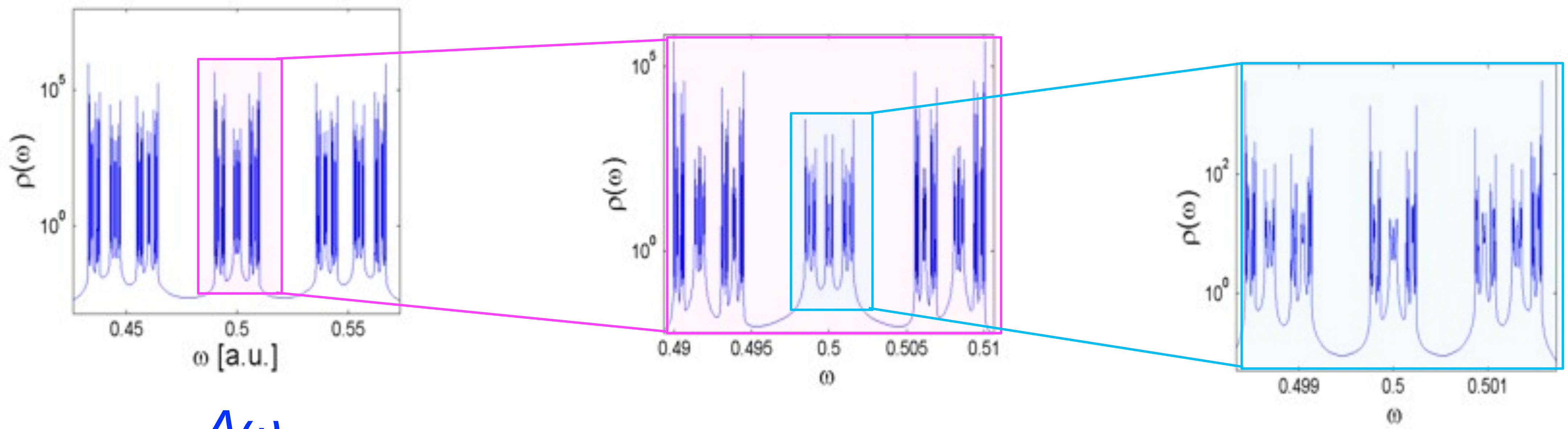

$\longrightarrow \Delta\omega \longleftarrow$

Counting function: $N_\omega(\Delta\omega) \equiv \int_{\omega}^{\omega+\Delta\omega} \rho(\omega')\,d\omega'$ = (# of states in $[\omega, \omega+\Delta\omega]$)

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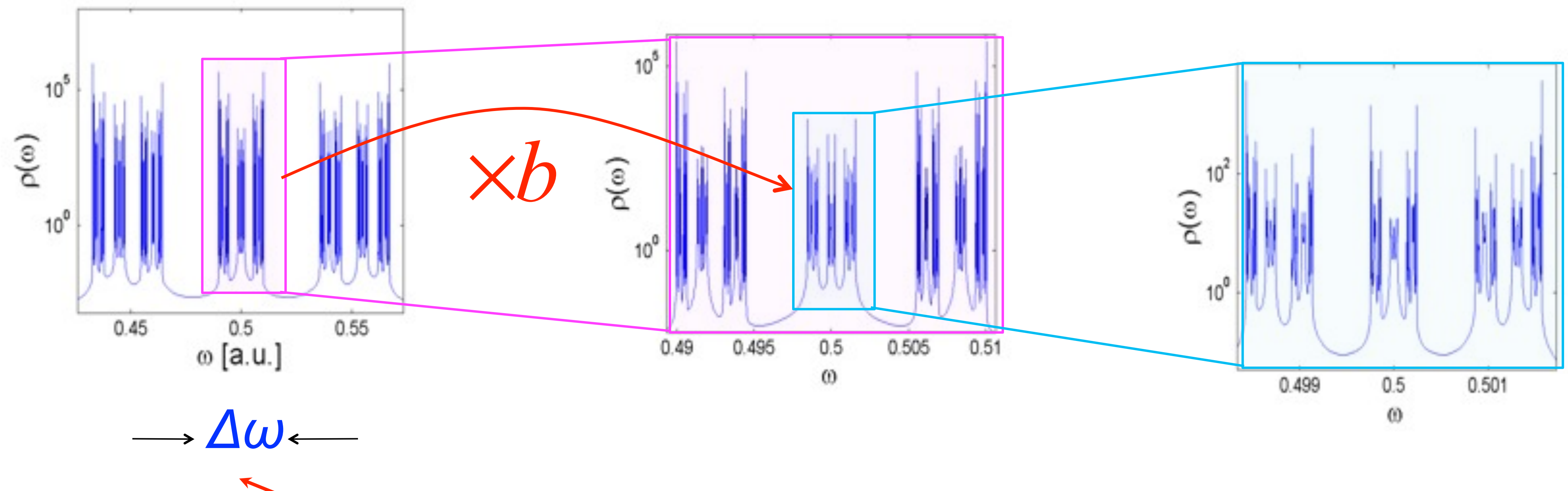

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b, a - fixed scaling factors

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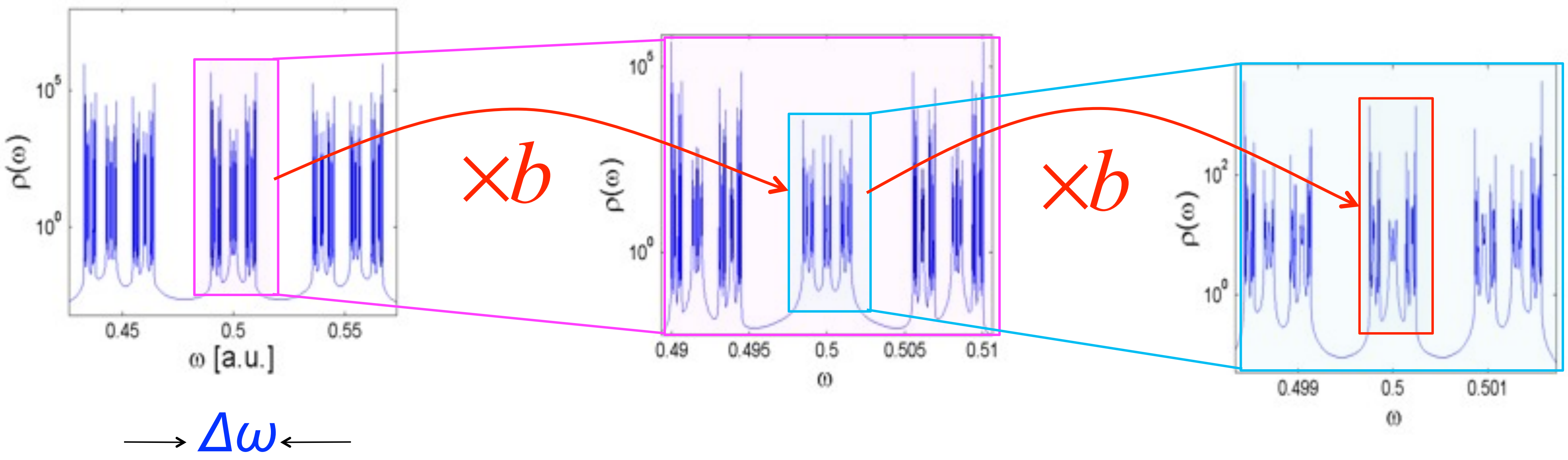

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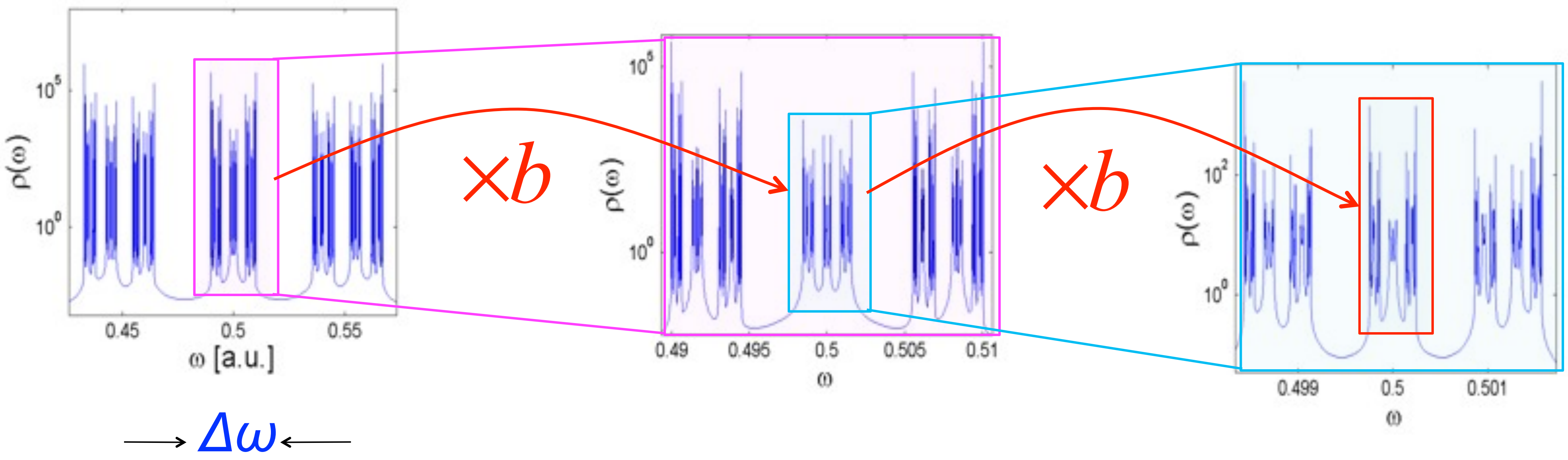

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$$N_\omega(b^p \Delta\omega) = a^p N_\omega(\Delta\omega)\,, \quad p \in \mathbb{Z}$$

b, a - fixed scaling factors

⇒ **Discrete scaling symmetry**

Testing the discrete scaling symmetry

Scaling equation

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has the following general solution (dimensionless ω):

$$N_\omega(\Delta\omega) = (\Delta\omega)^\alpha \times \left(\cdots\right) \quad , \quad \alpha = \frac{\ln a}{\ln b}$$

$0 \le \alpha \le 1$ - fractal exponent (absolutely continuous : $\alpha = 1$, pure-point : $\alpha = 0$)

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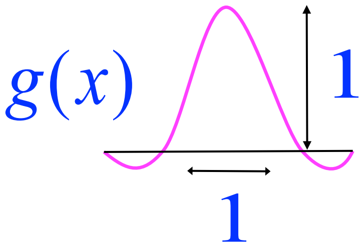
Similarly for the convolution of $\rho(\omega)$ with a window function



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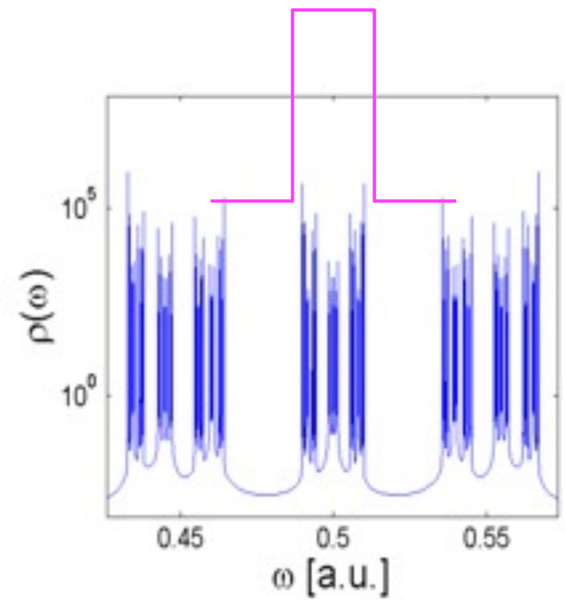

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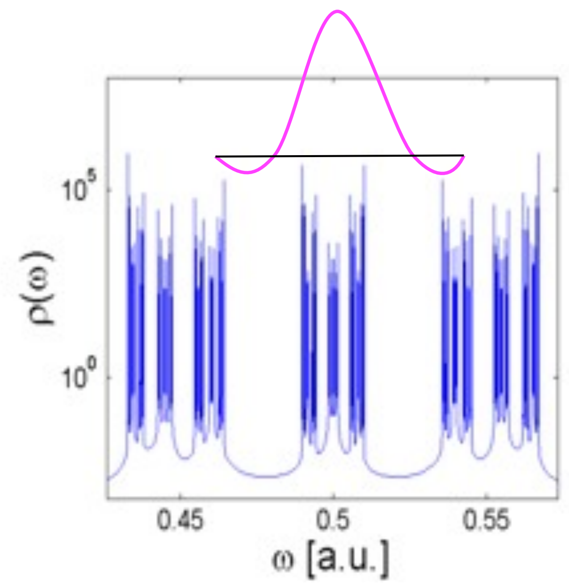

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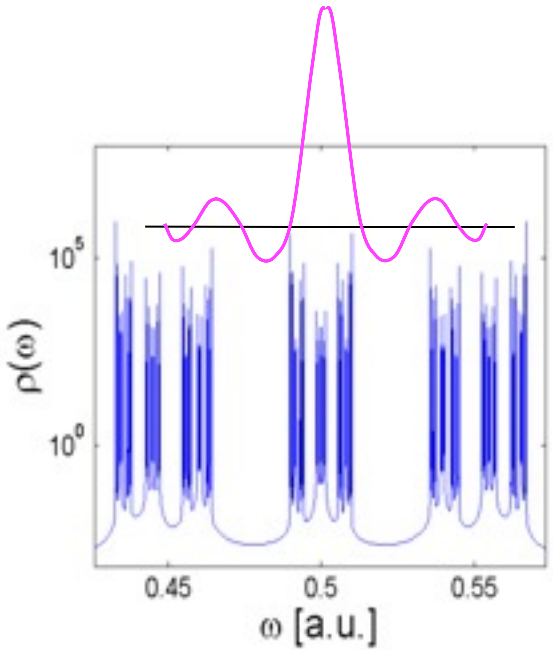
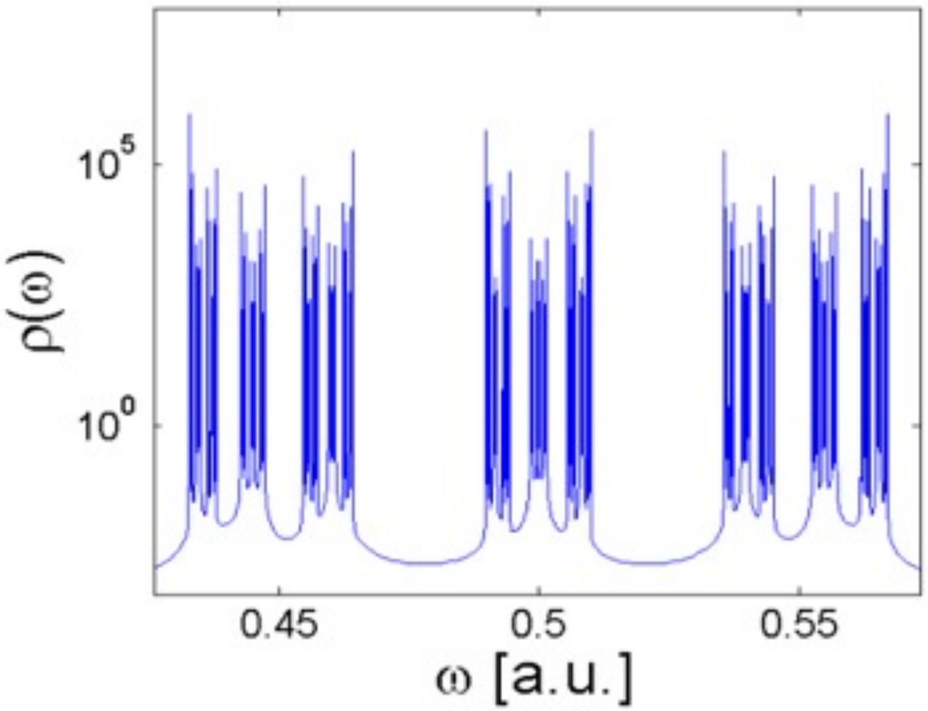
(Ghez and Vaienti, '89: the wavelet transform of fractal measures)



Testing the discrete scaling symmetry - an example

A quasi-periodic dielectric stack



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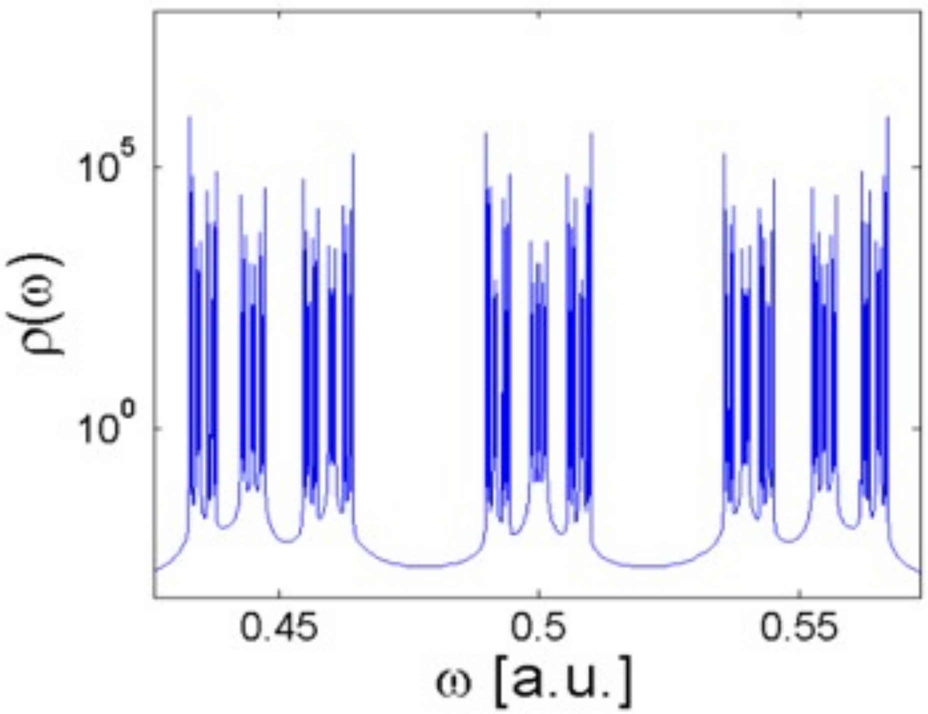
$$N_{\omega}^{(g)}(\Delta\omega) \equiv \int g\left(\frac{\omega' - \omega}{\Delta\omega}\right)\rho(\omega')\,d\omega' \stackrel{?}{=} (\Delta\omega)^{\alpha} \times F_g\left(\frac{\ln|\Delta\omega|}{\ln b}\right),$$



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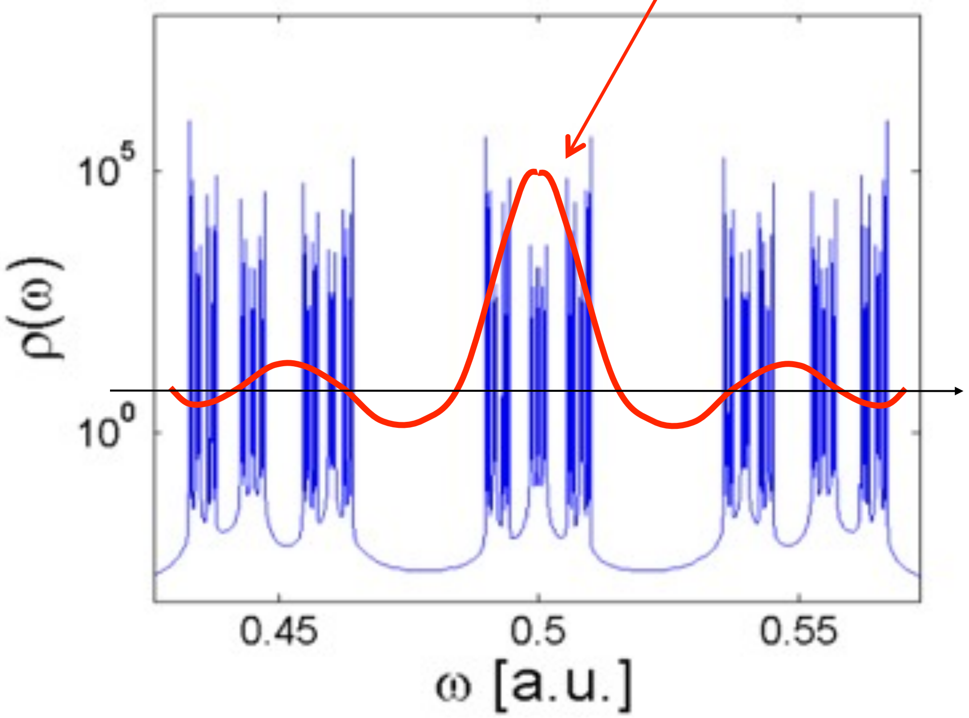

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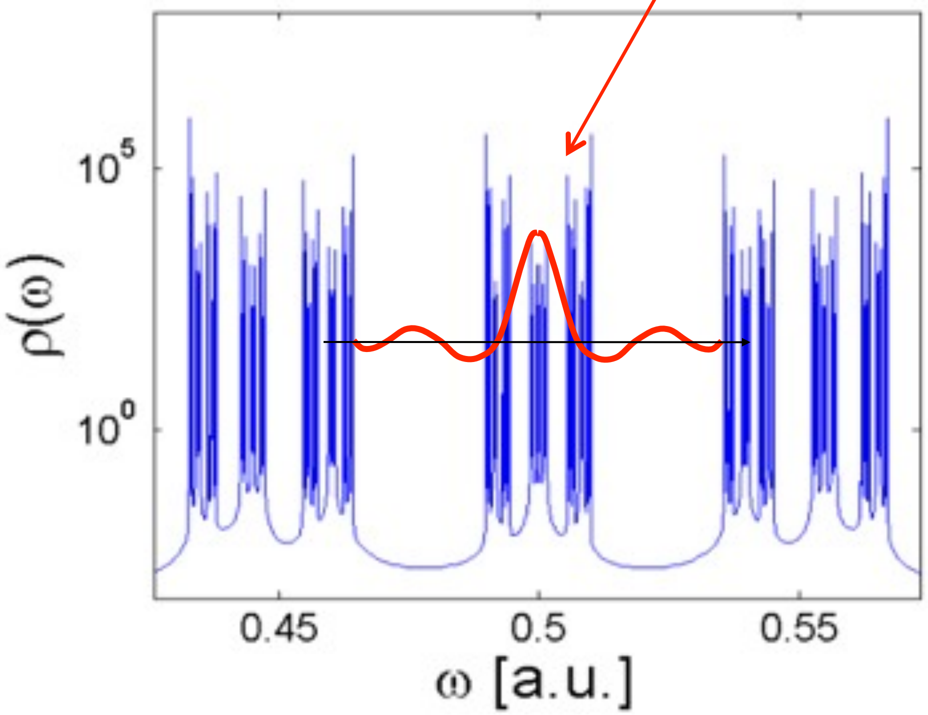

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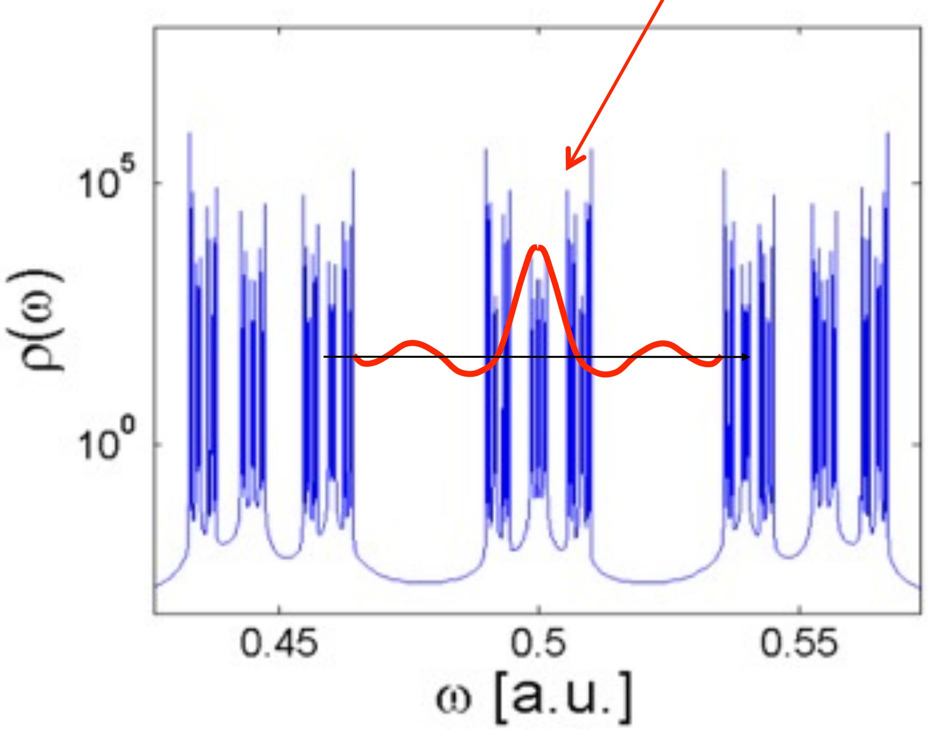

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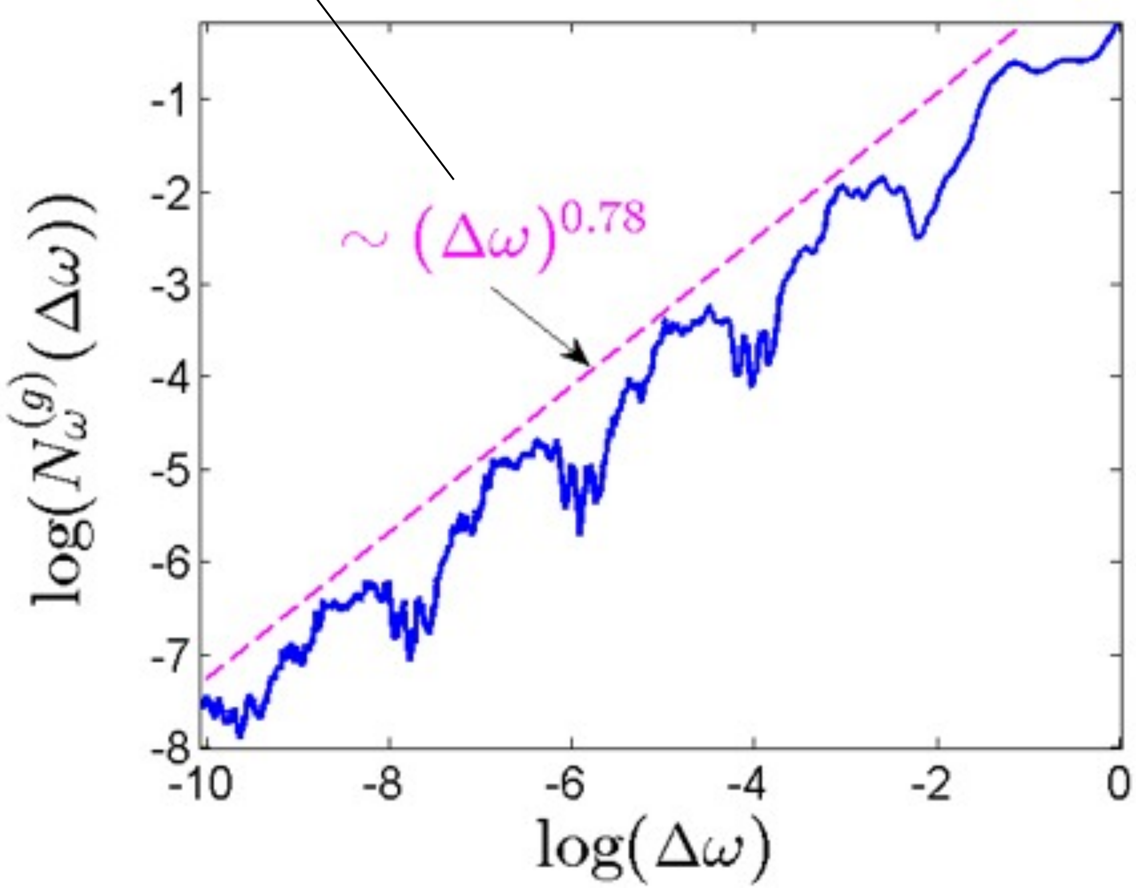

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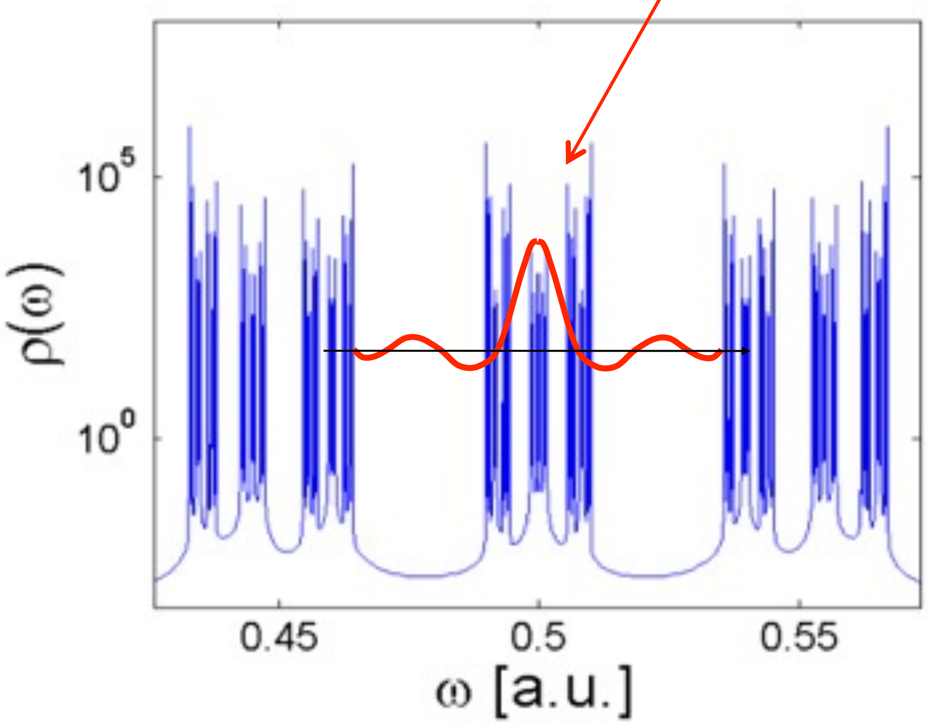

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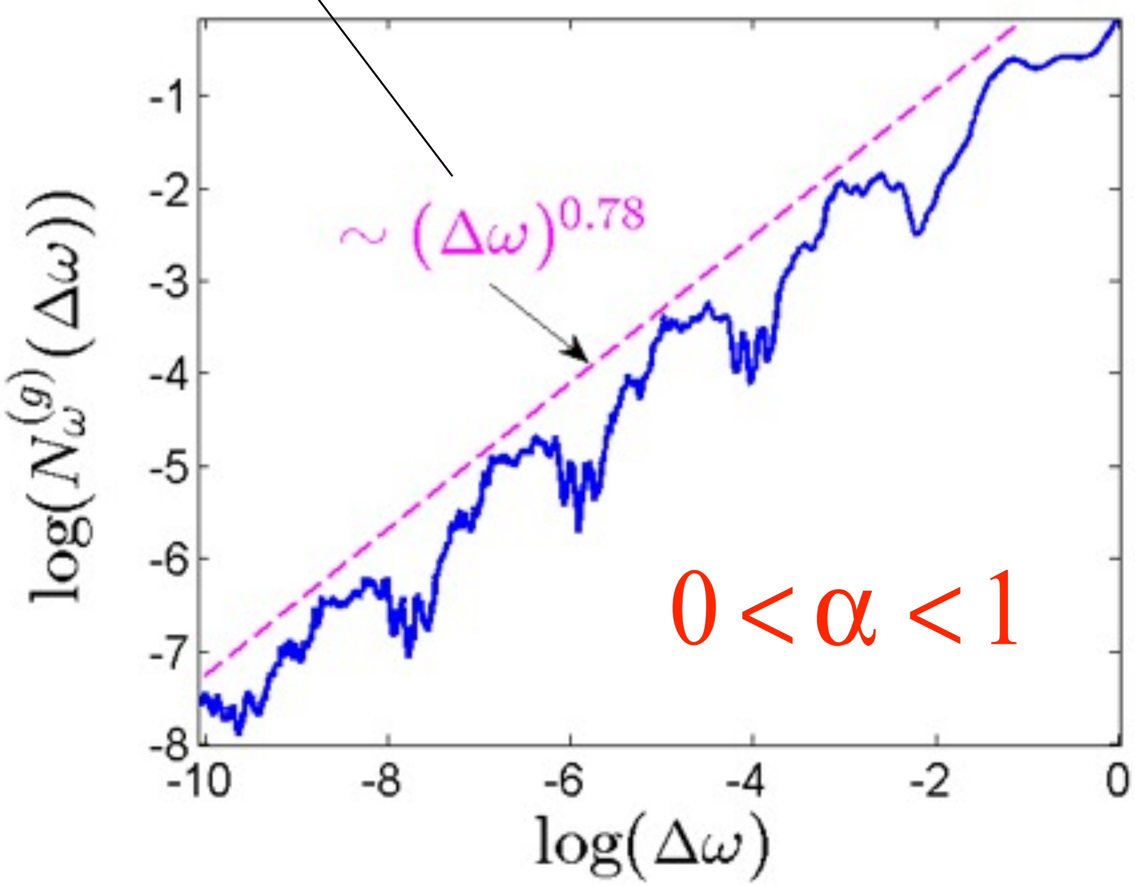
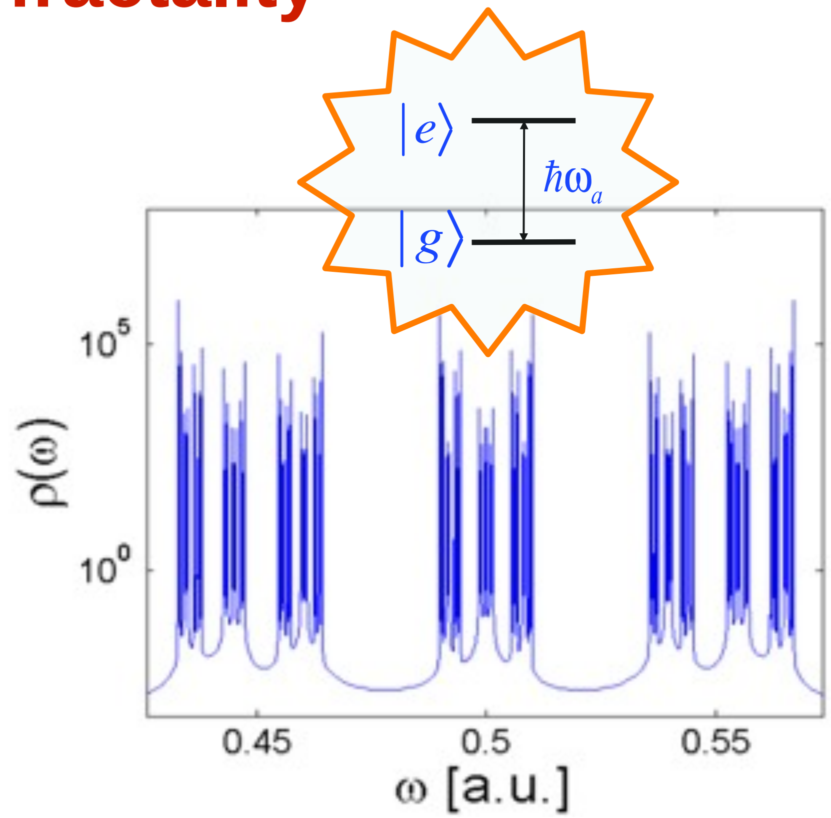


$\alpha = \alpha(n_A, n_B) = 0.777$ (Kohmoto *et. al.*, '87)

Spontaneous emission and vacuum fractality



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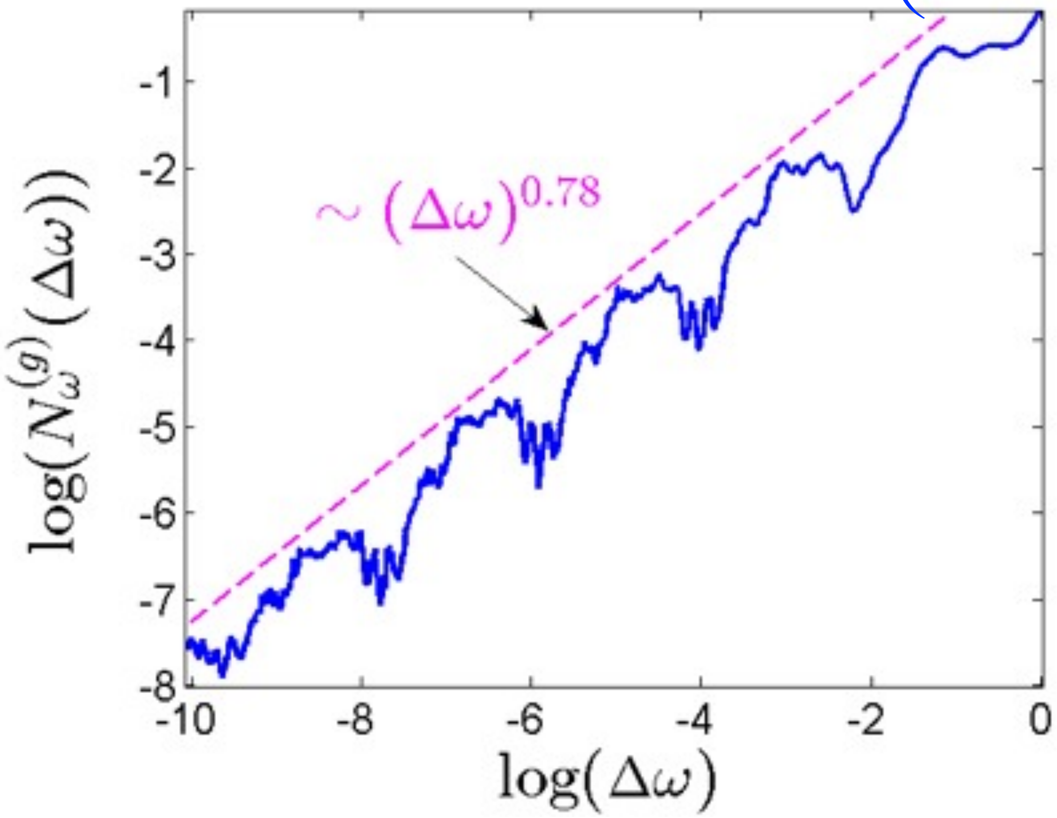

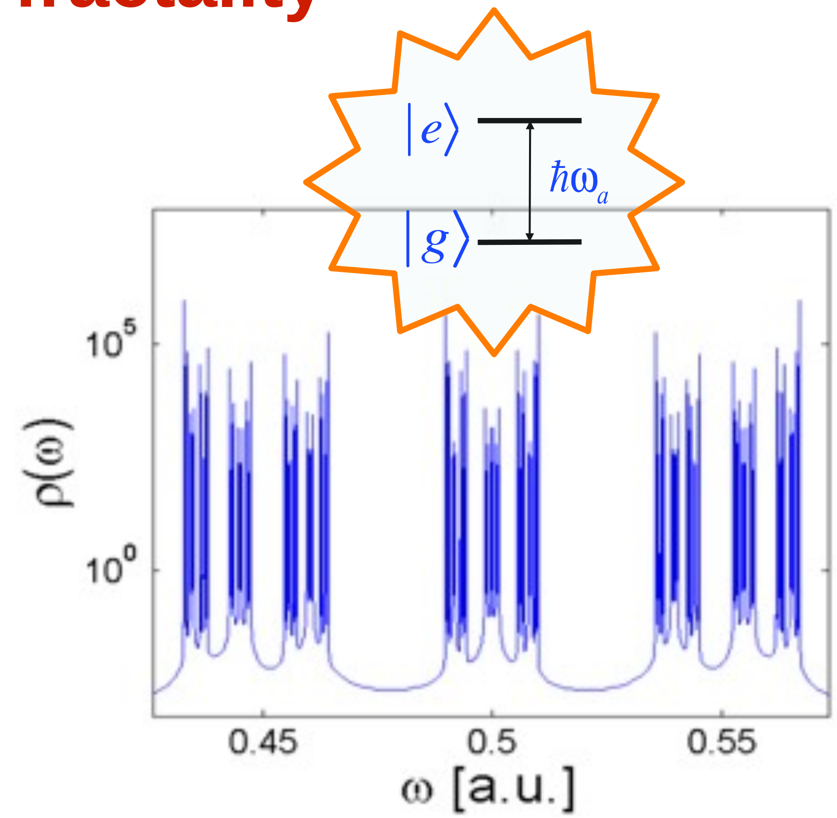

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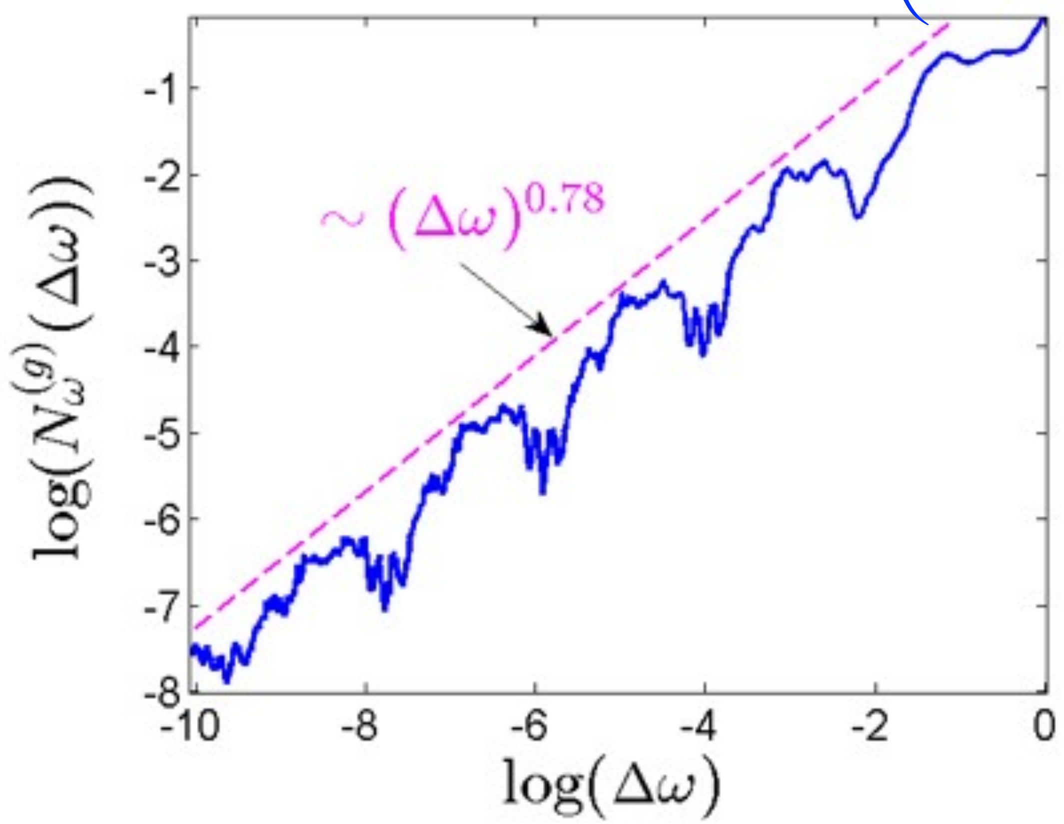
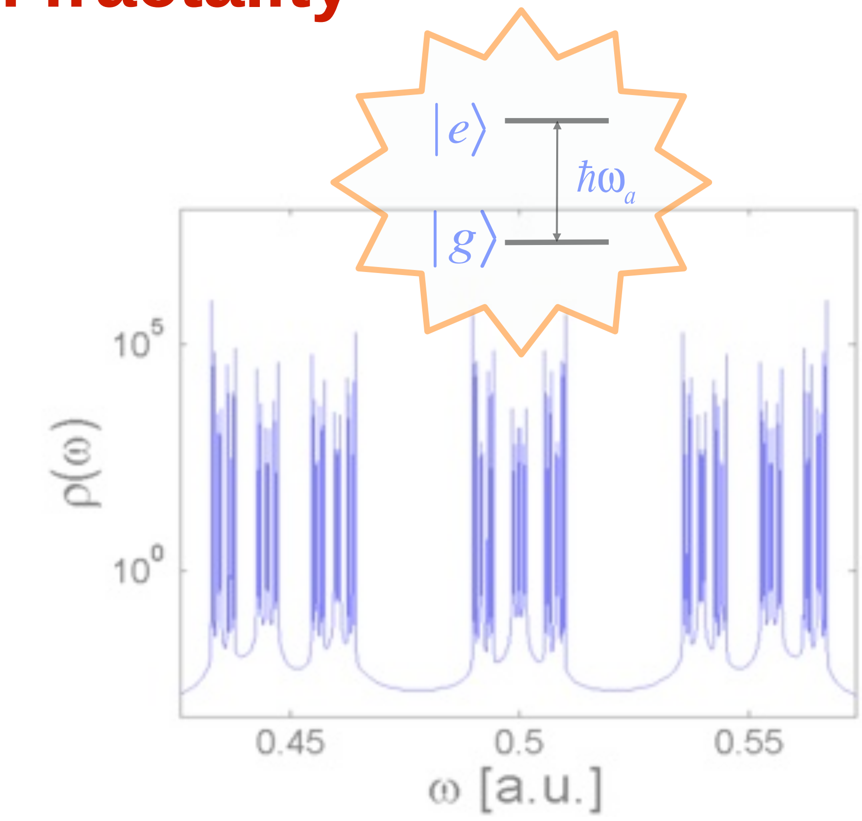
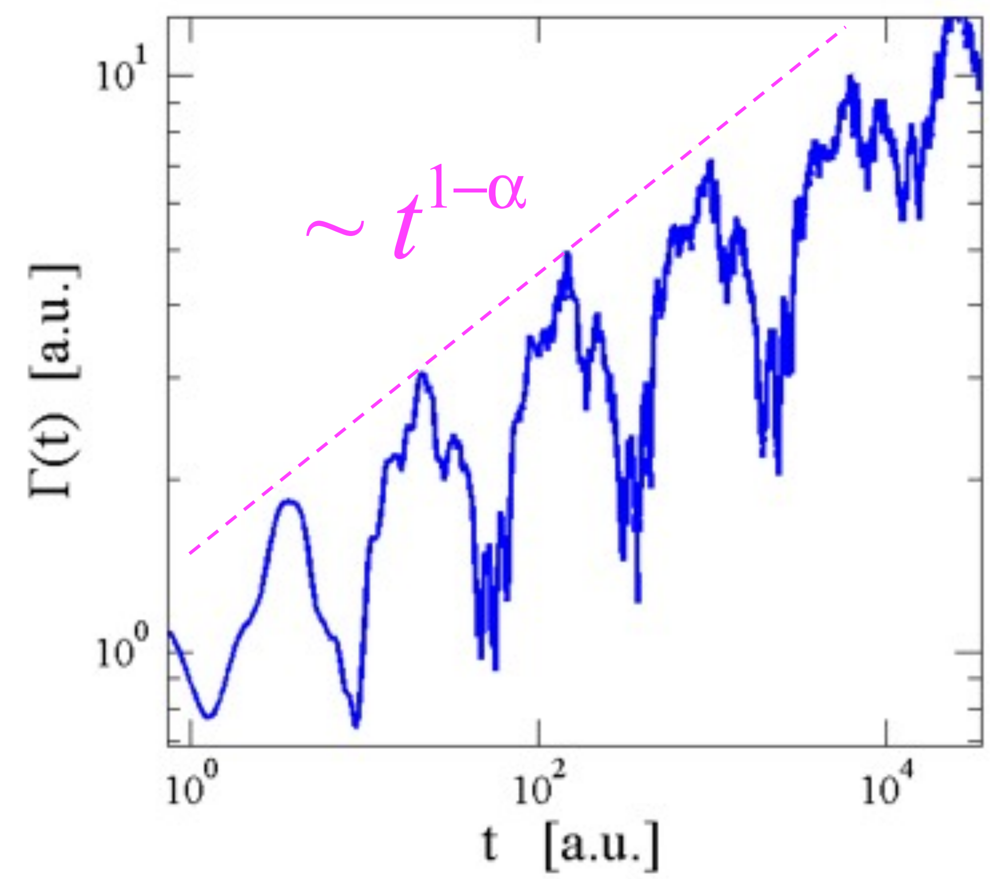




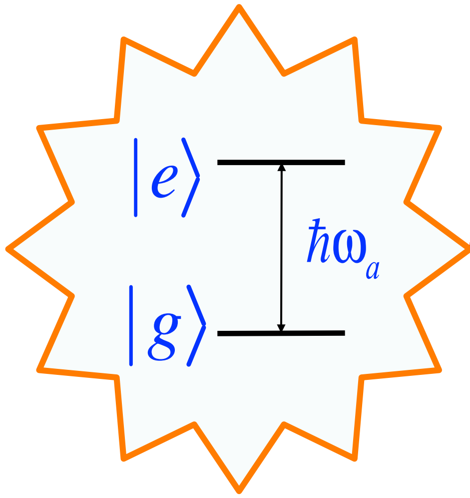
$$\Delta\omega \sim t^{-1}$$

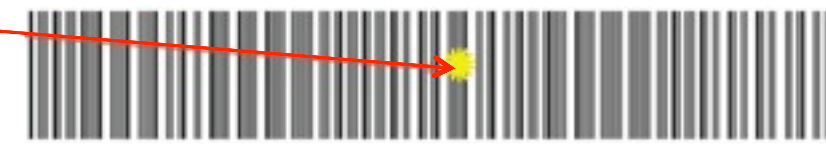


Differential decay rate

(at small times) $\Gamma(t) = \frac{dp_e(t)}{dt}$

Two-level atom coupled to a continuum of states





$$H_{Int} = \sum_k V_k a_k^\dagger |g\rangle\langle e| + h.c.$$

$$V_k \sim E_k(r_a)$$

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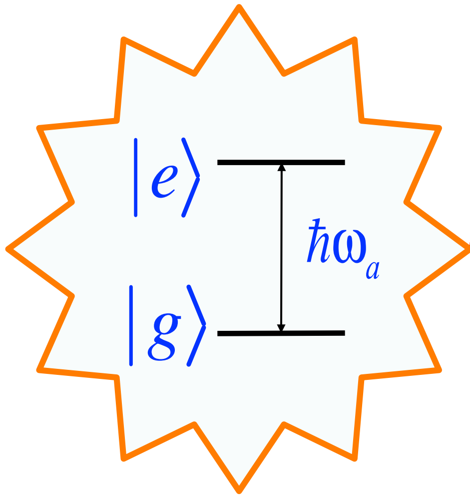

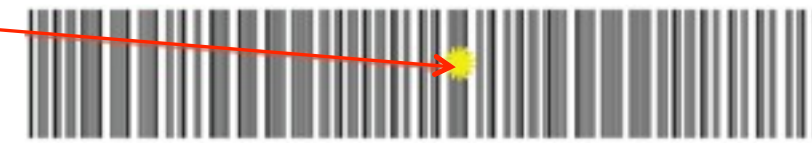

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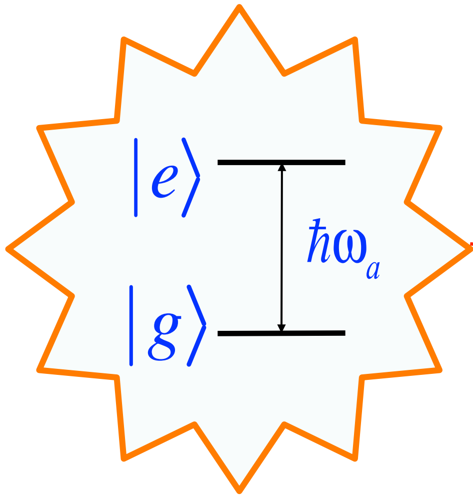

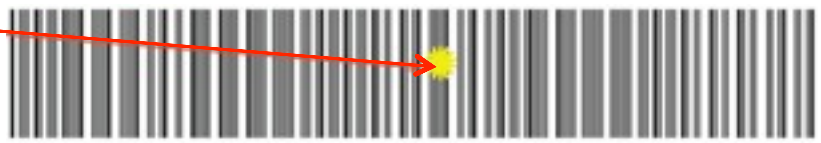

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density of photonic modes

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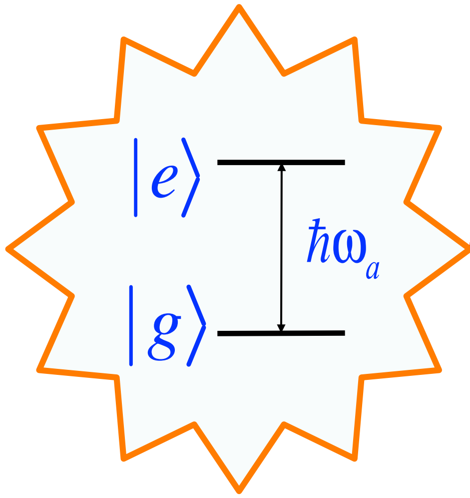

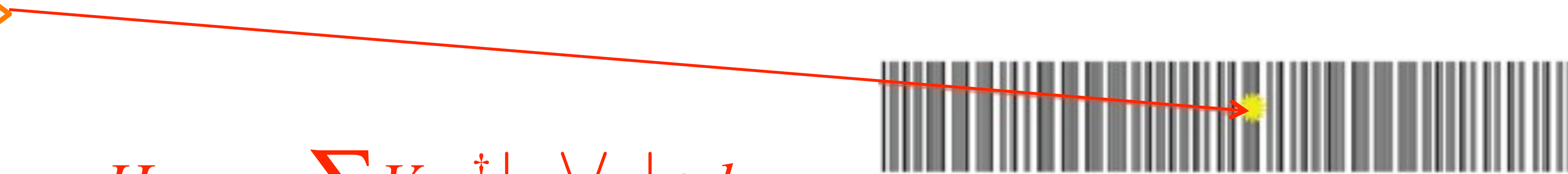

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density of photonic modes

$p_e(t) = |\alpha(t)|^2$ - the excited state probability

Short time limit – the Fermi golden rule revisited

Short-time limit

A standard perturbative treatment:

For short times, such that $\alpha(t) \approx \alpha(0) = 1$

the excited state probability is

$$p_e(t) \approx 1 - \int_0^t \Gamma_e(t')dt',$$

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where the differential decay rate $\Gamma_e(t)$ is given by the well known expression:

$$\Gamma_e(t) = \frac{2}{\hbar^2}\int dk\rho(k)\,|V_k|^2 \frac{\sin(\omega_k - \omega_a)t}{(\omega_k - \omega_a)}$$

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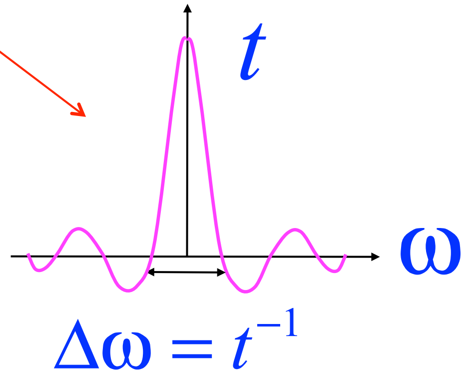

Fermi golden rule

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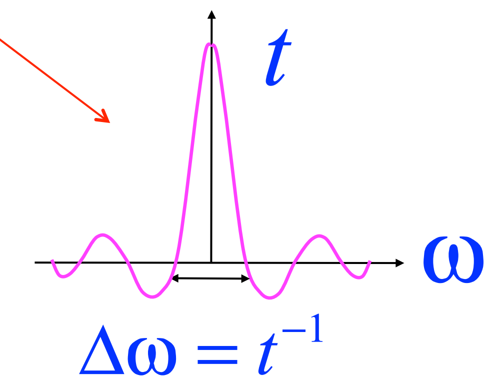

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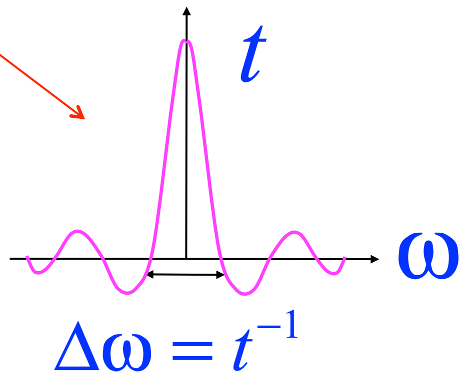

Valid for smooth spectrum + long times

$$\Gamma_e(t) = \frac{2}{\hbar^2}\int dk\,\rho(k)\,|V_k|^2 \pi\delta(\omega_k-\omega_a) = const = \Gamma_e$$

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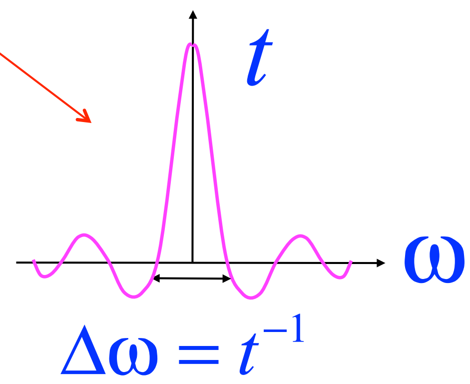

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This Γ_e coincides with the exponential decay rate (Wigner-Weisskopf):

$$p_e(t) \approx 1 - \Gamma_e t \quad \longleftrightarrow \quad p_e(t) = e^{-\Gamma_e t}$$

Short time limit - fractal spectrum

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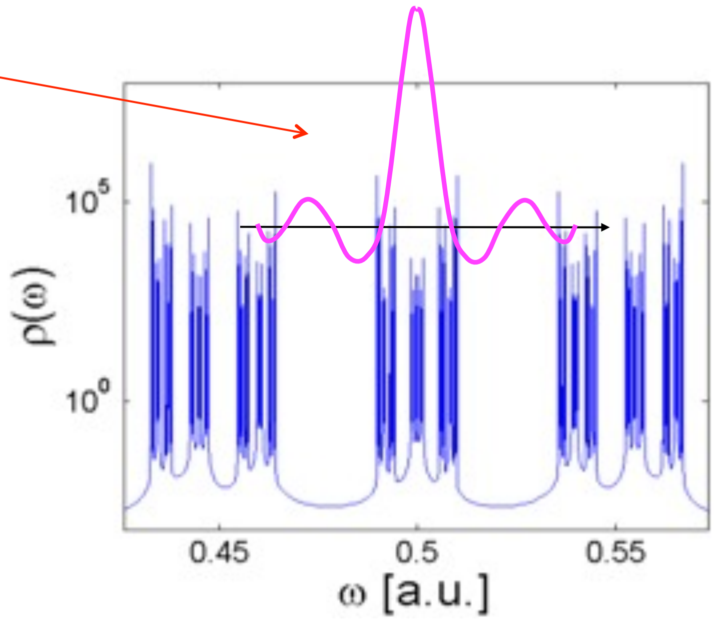

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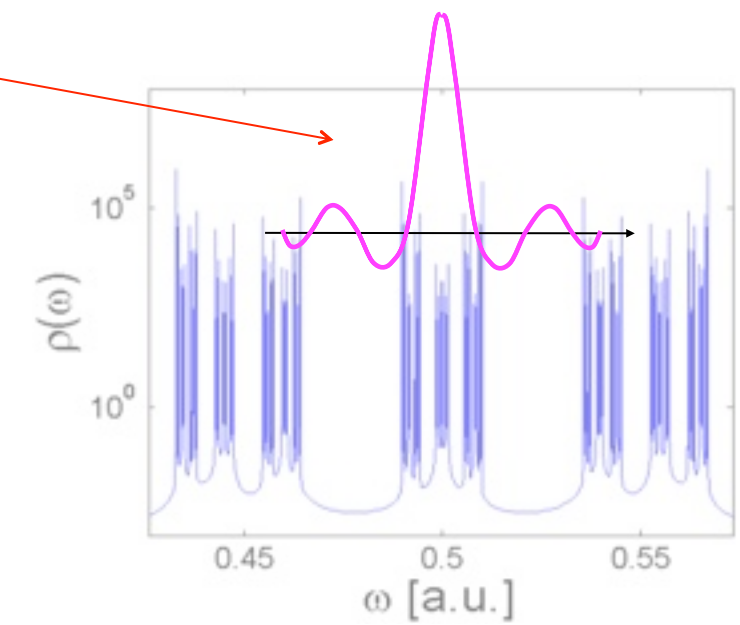

$$\frac{\sin(\omega_k - \omega_a)t}{(\omega_k - \omega_a)} \approx \pi\delta(\omega_k - \omega_a)$$

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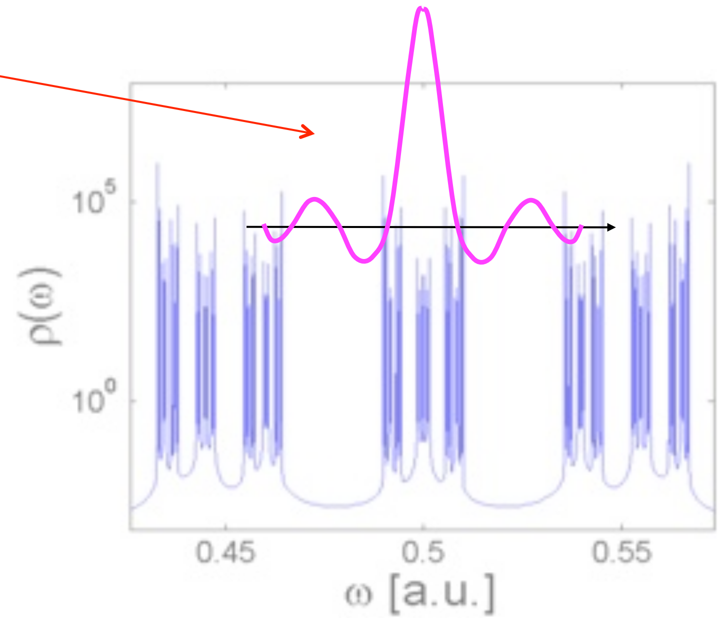

Recall that the counting function satisfies

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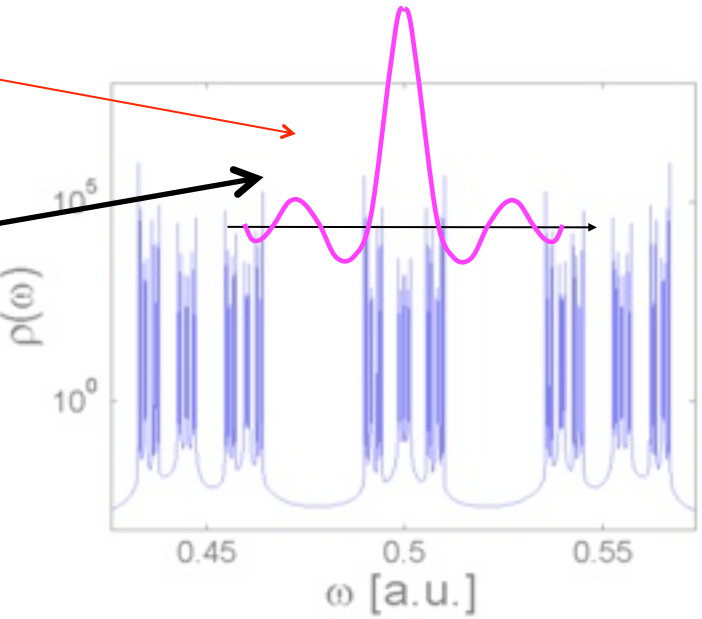

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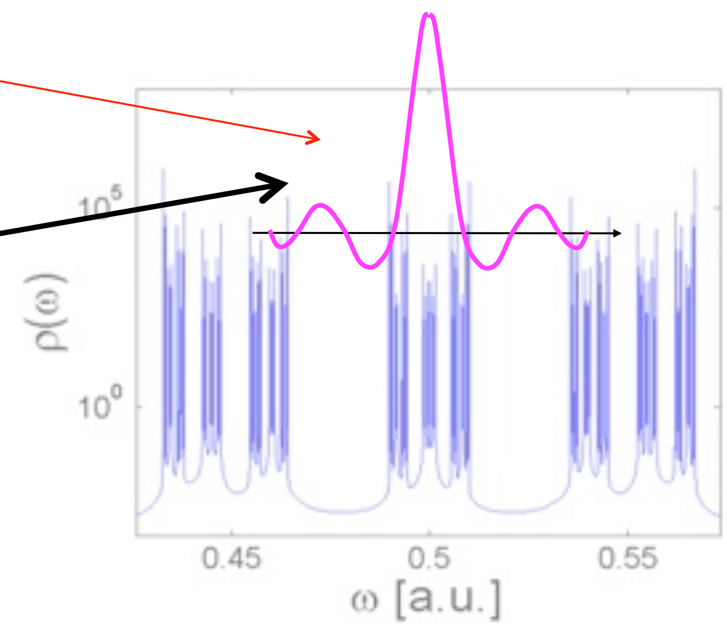

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We immediately conclude that the general form of $\Gamma_e(t)$ is:

$$\Gamma_e(t) = \tau^{-1}\times\left(\frac{t}{\tau}\right)^{1-\alpha}\times F\left(\frac{\ln(t/t_0)}{\ln b}\right), \qquad F(x+1) = F(x),$$

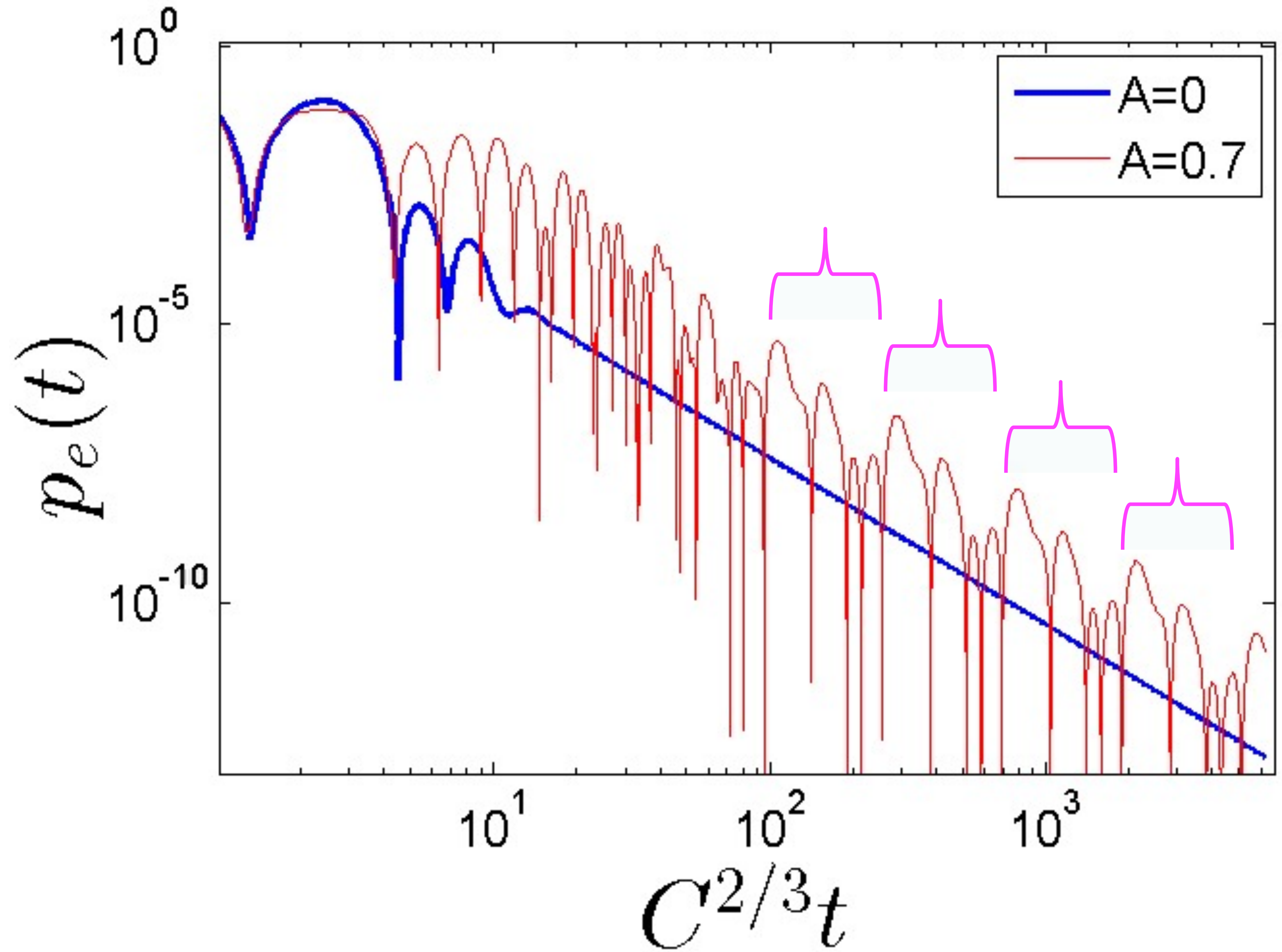
where

$0 \le \alpha \le 1,\quad b$ - fractal exponent and scaling factor of the spectrum

τ, t_0 - time scales, specific to the considered problem.

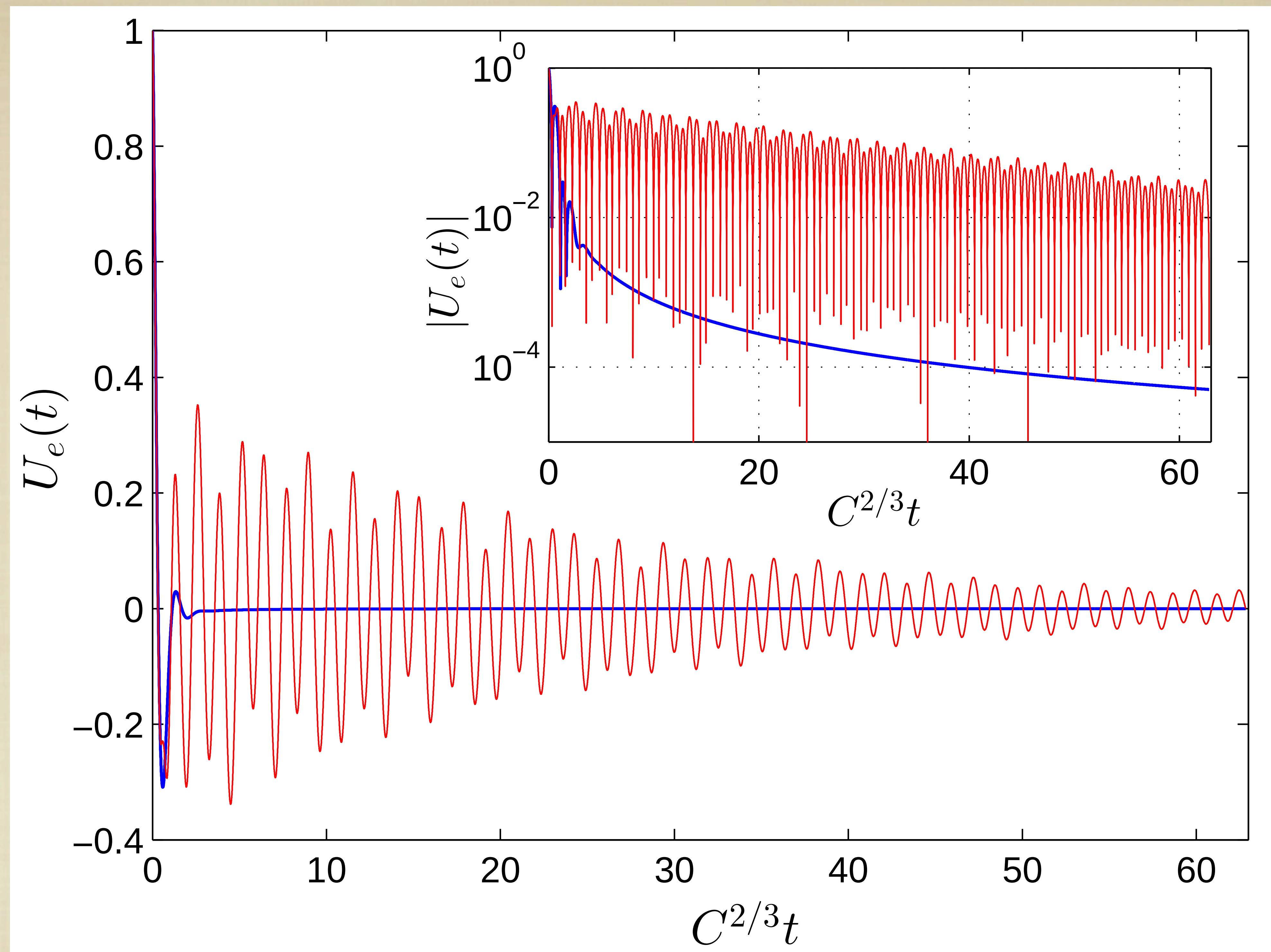
Beyond the short time regime- Strong coupling and Inhibition of spontaneous emission

A toy model



$$\alpha = \frac{1}{2}$$

$$\rho_L(k, r_a) \sim \overline{|V_k|^2}\rho(\omega) = \frac{C}{|\omega - \omega_u|^{1-\alpha}}\left[1 + A \times \cos\left(\frac{\ln|\omega - \omega_u|}{\ln b}\right)\right],$$



Strong coupling - non perturbative solution

Experimental study of a fractal energy spectrum :

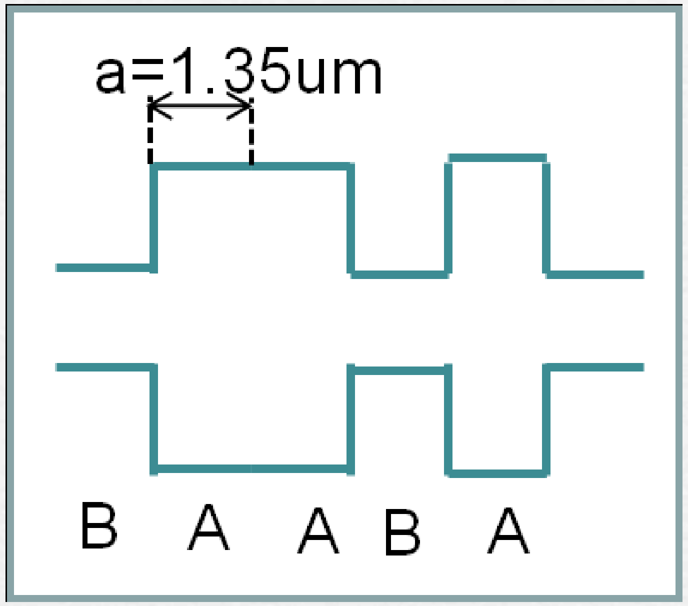
Coherent polaritons gas in a Fibonacci quasi-periodic potential

D. Tanese, J. Bloch, E. Gurevich, E.A. 2013.

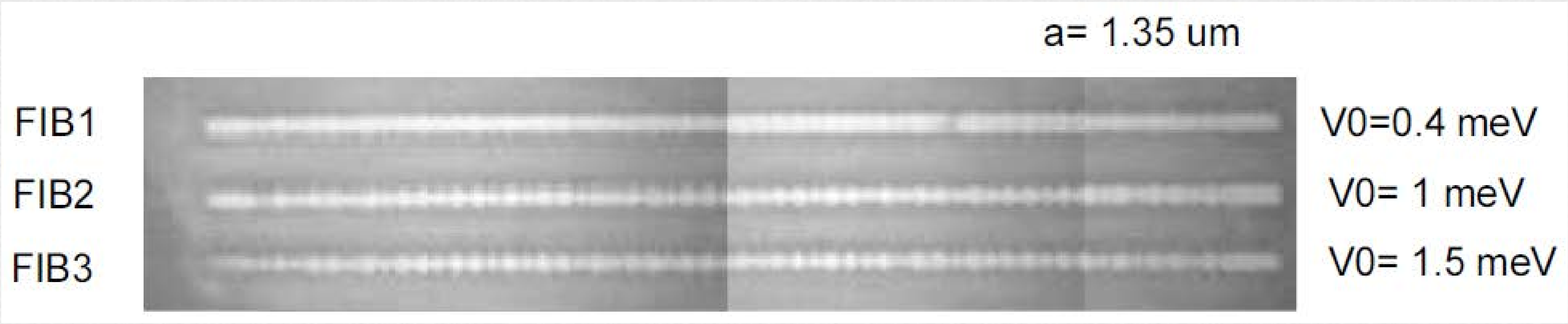
The Fibonacci problem has a long and rich (theoretical and experimental) history.

(Kohmoto,Luck, Gellerman, Damanik, Bellissard,Simon,...)

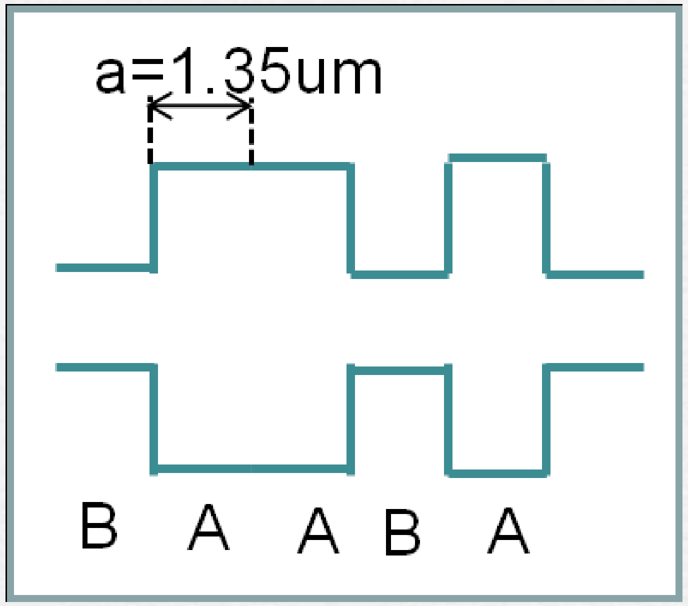
Our purpose here is to propose a quantitative description of fractal properties in order to use fractals/singular continuous systems as useful simulating tools



Number of letters of a sequence S_j is the Fibonacci number F_j so that $F_j = F_{j-1} + F_{j-2}$



(193 letters)



Fibonacci sequence: $S_{j\geq 2} = \left[S_{j-1} S_{j-2} \right], \quad S_0 = B, \quad S_1 = A$

A→AB→ABA→ABAAB→ABAABABA→...

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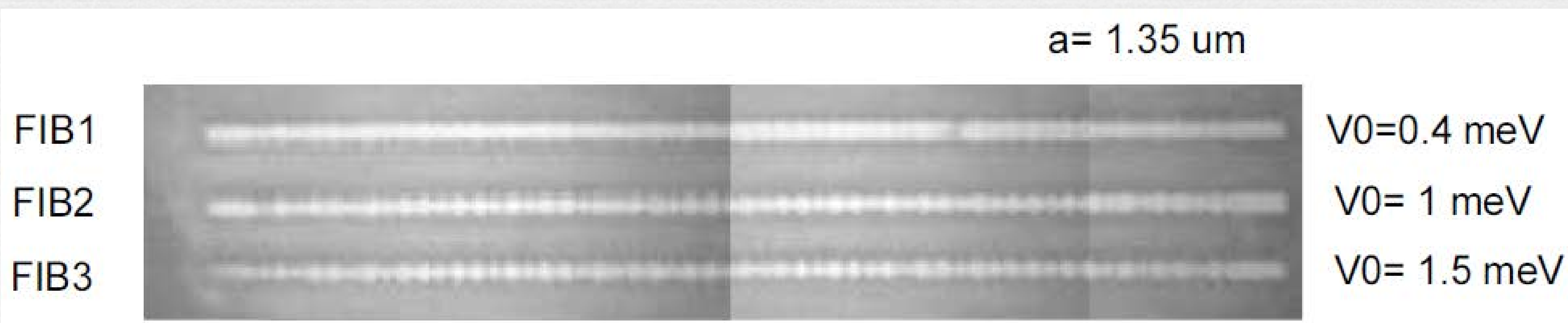

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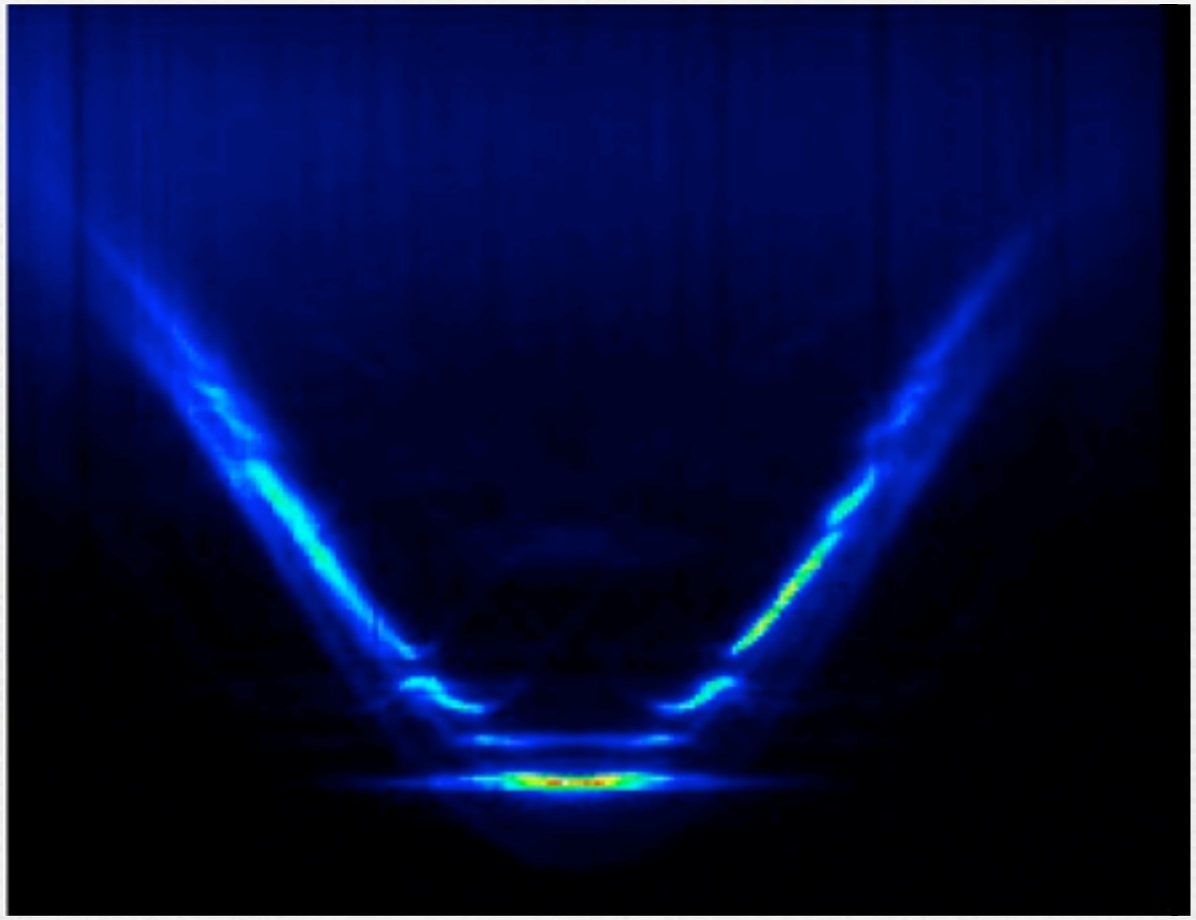
Measure of spectral function $E(k)$ intensity maps



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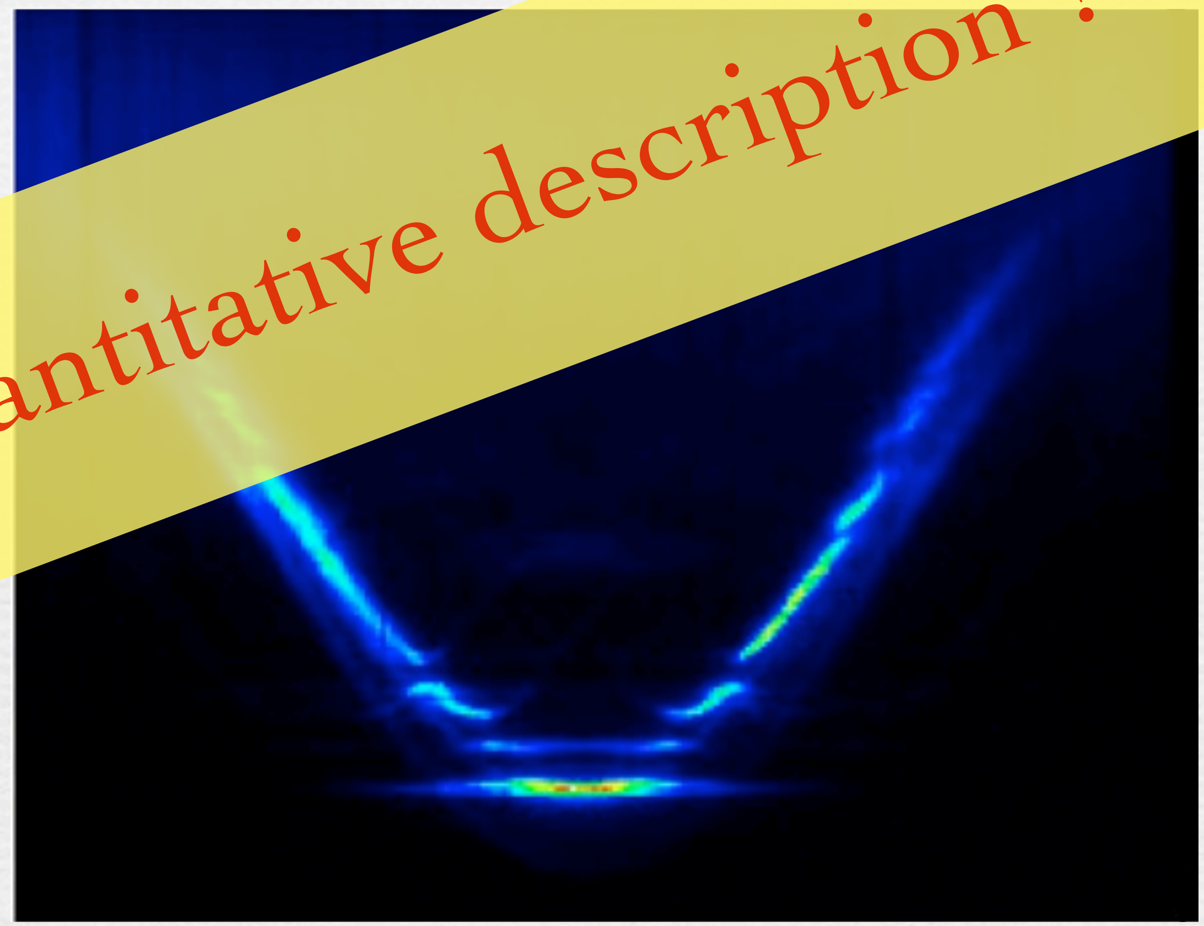

Quantitative description !

Effective 1D model

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dx^2}+V(x)\right]\psi(x)=E\psi(x)$$

where

$$V(x)=u_b(x)\times\left[\chi(\sigma x)\sum_n \delta(x-na)\right]$$

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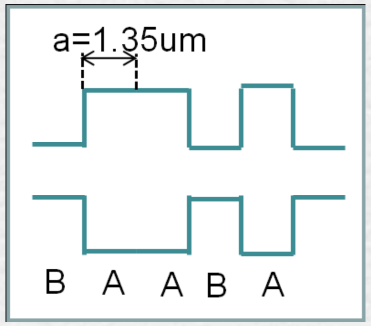
Shape of each letter



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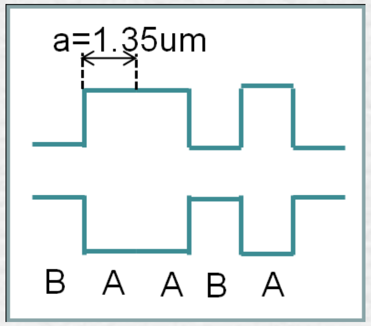

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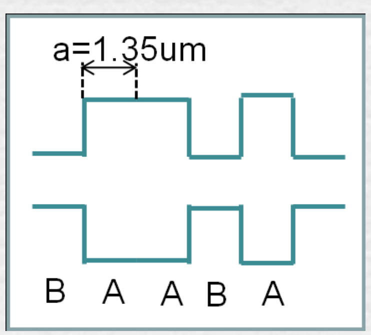
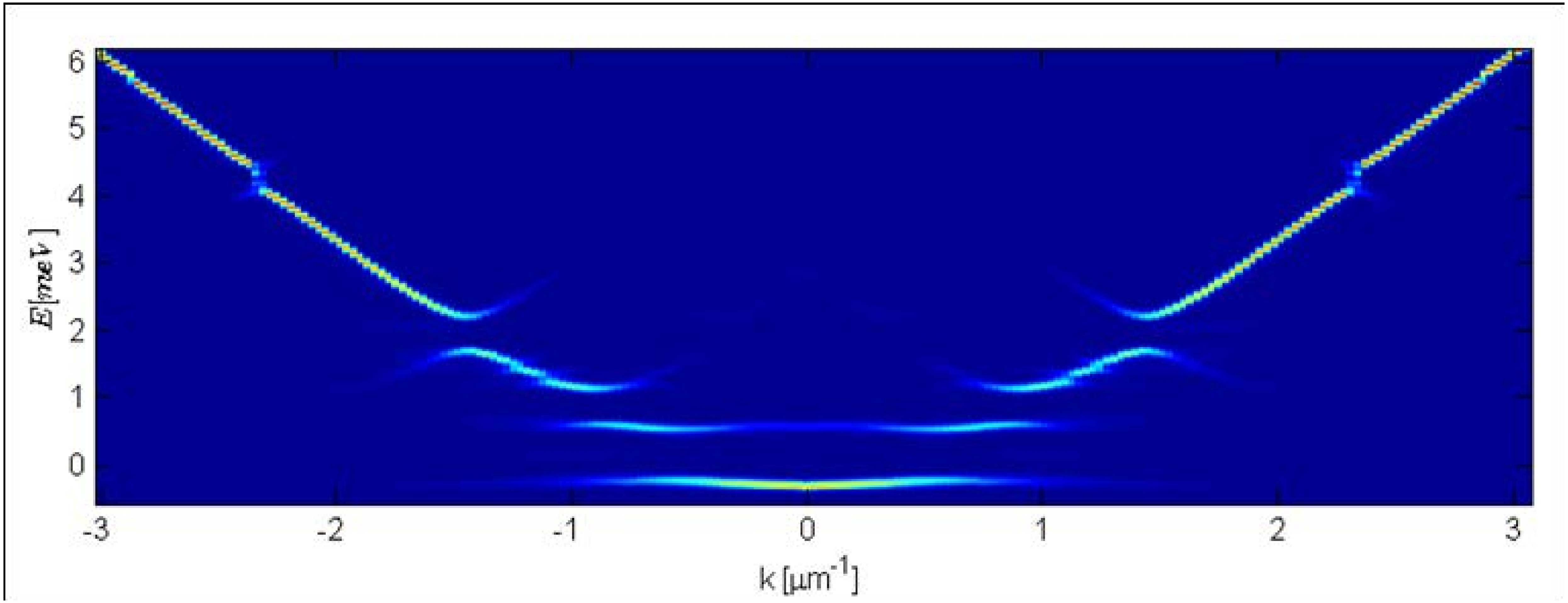
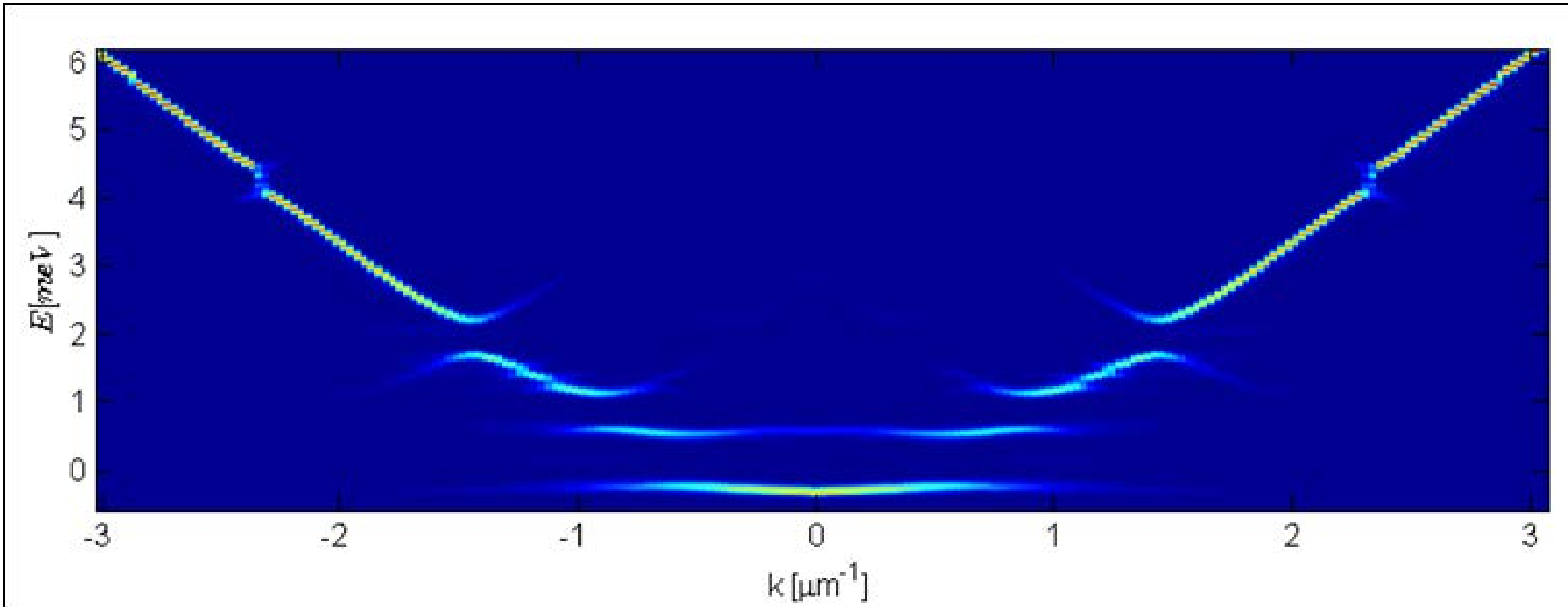


$\sigma = {\left(\sqrt{5}-1\right)}/{2}$ is the inverse golden mean

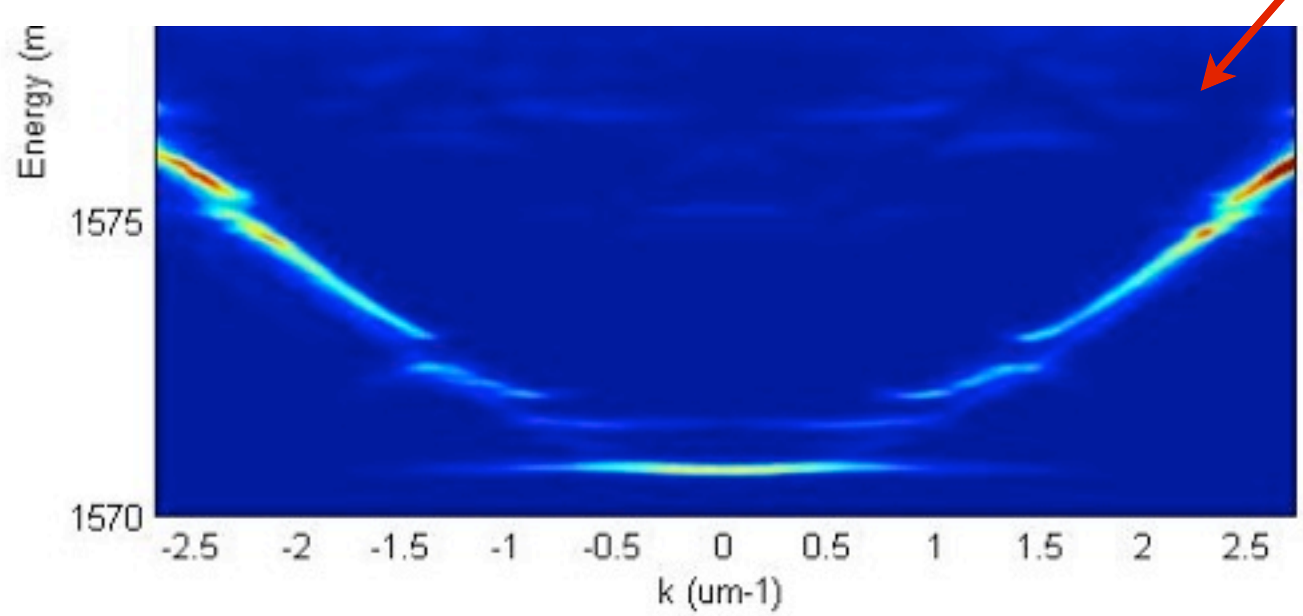
No fitting parameter except for the smoothness of $u_b(x)$



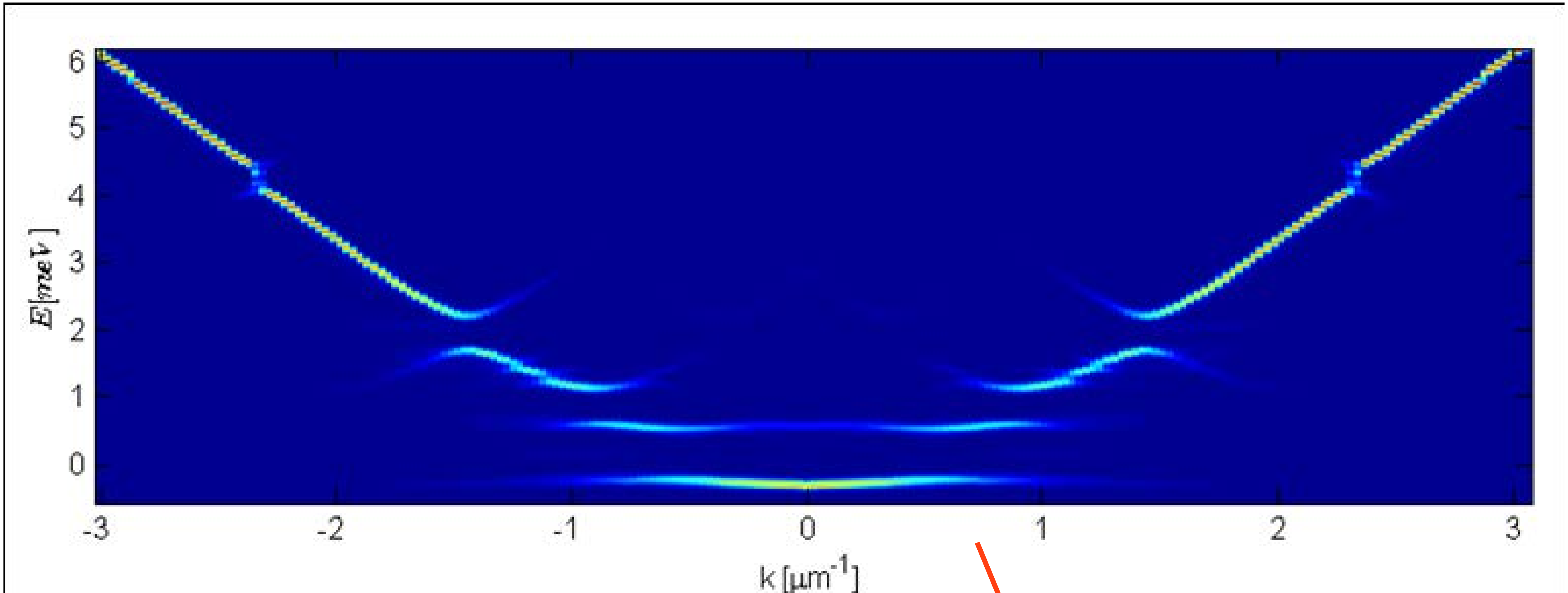
Spectral function $E(k)$ intensity maps (Numerics)



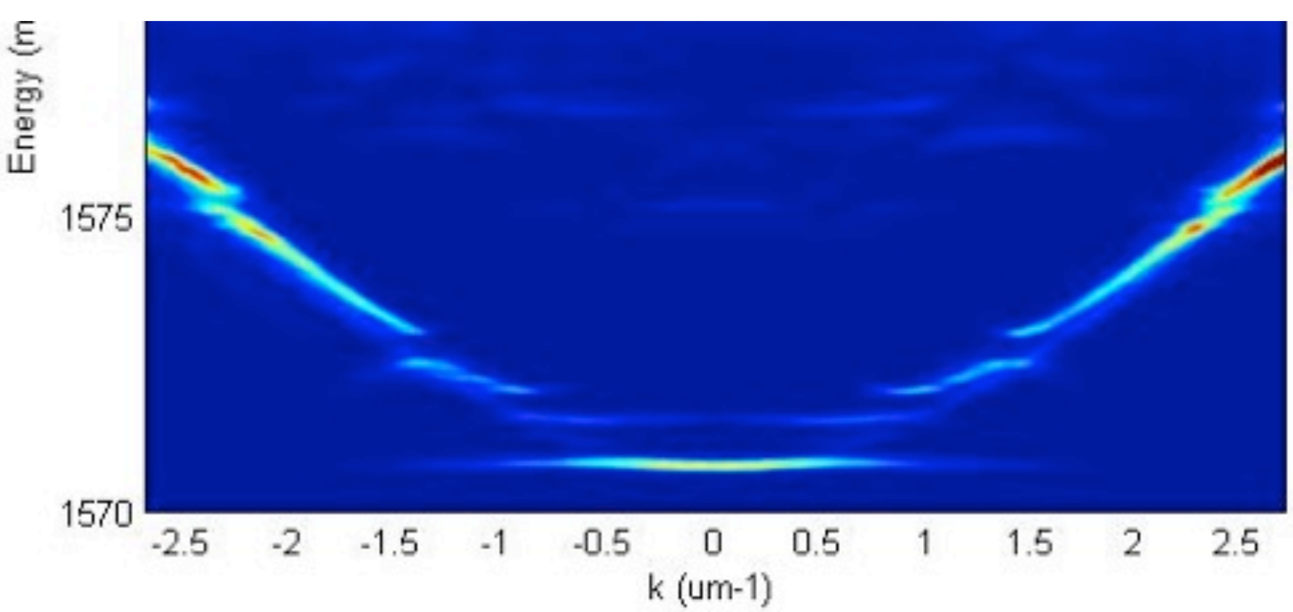
Spectral function $E(k)$ intensity maps (Experimental)



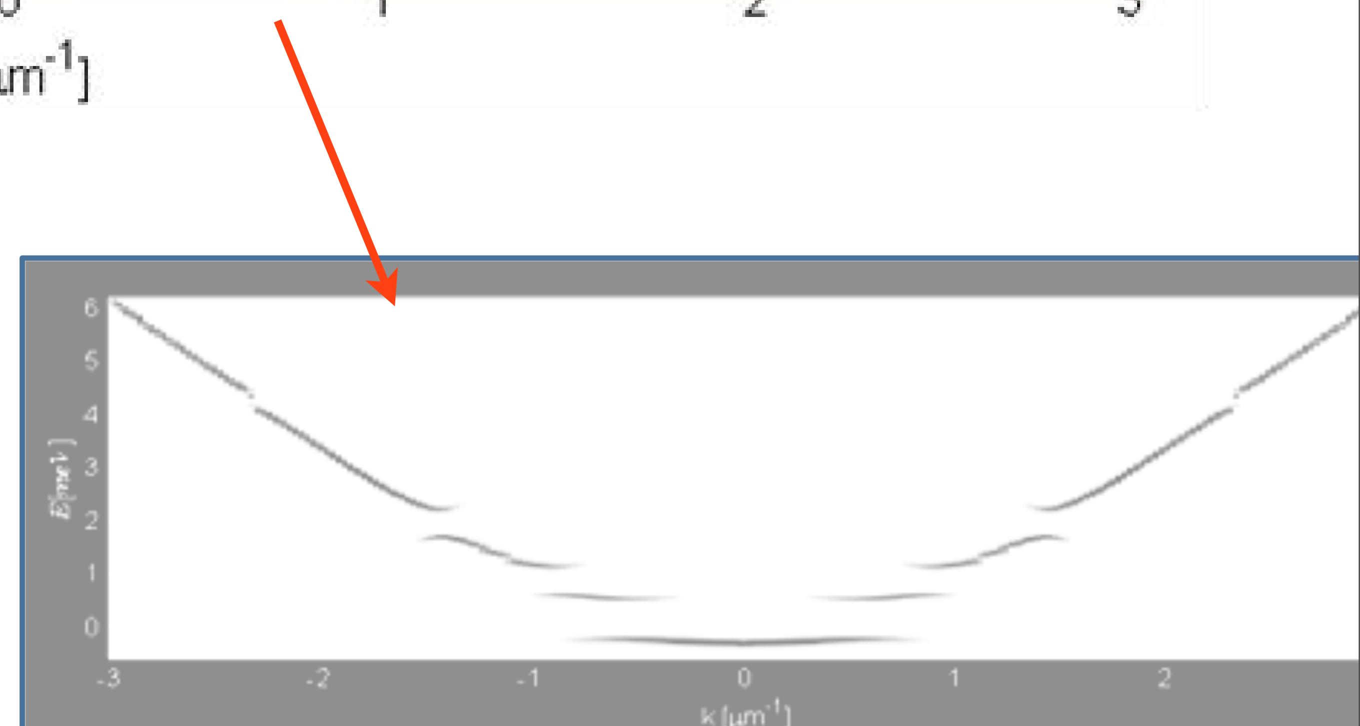
Comparison



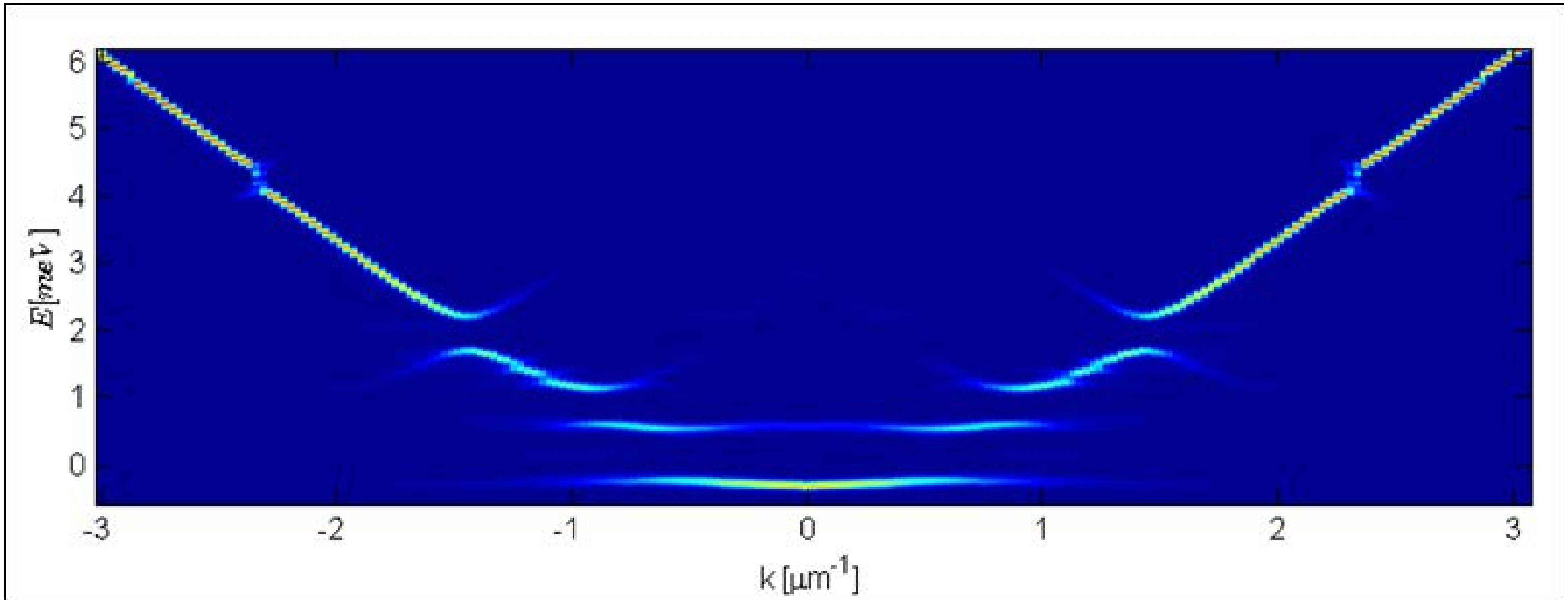
6
5
4
3
2
1
0
$E[meV]$
-3
-2
-1
0
1
2
3
k [µm-1]



Energy (m
1575
1570
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
k (um-1)



Comparison



6
5
4
3
2
1
0
E[meV]
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-1
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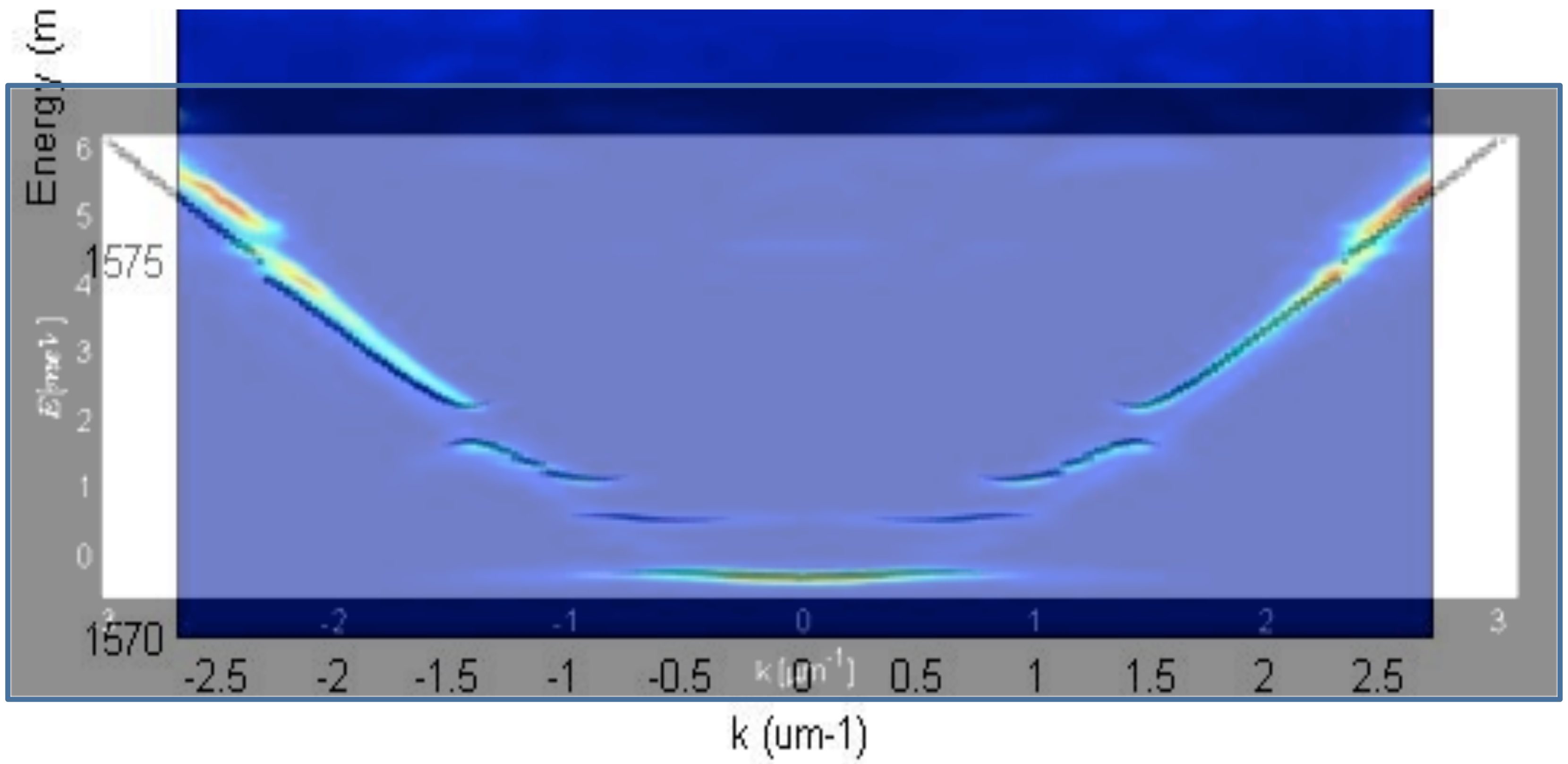
1
2
3
k [μm^-1]



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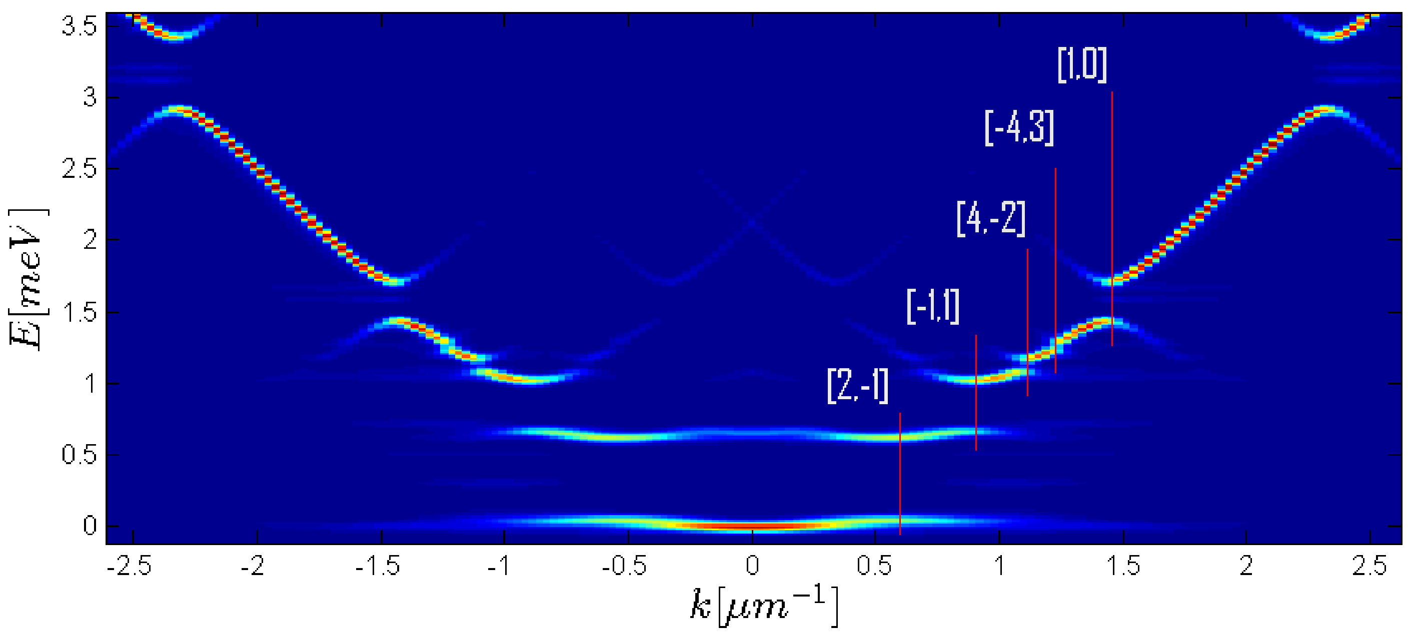
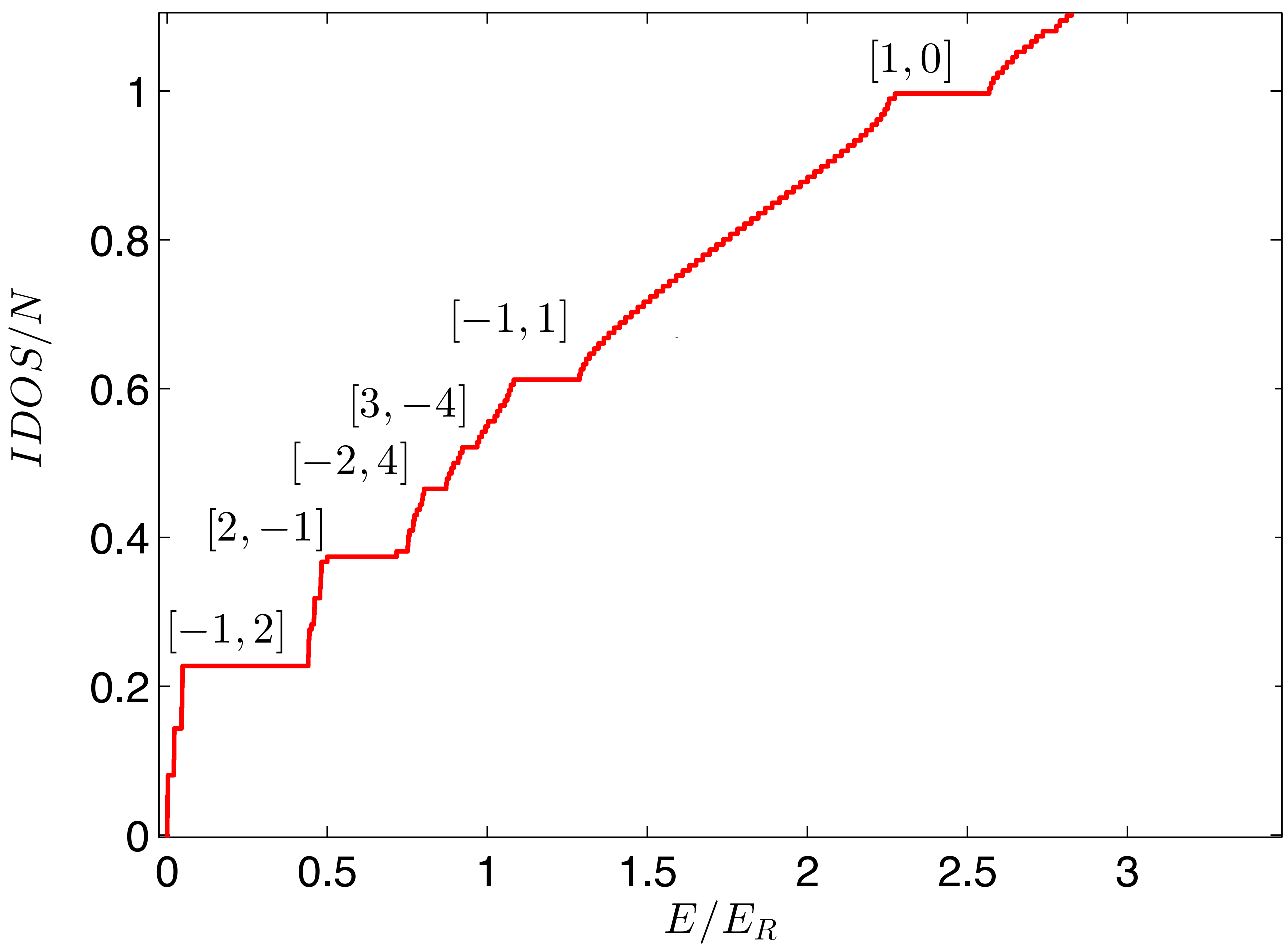
Labeling the gaps...





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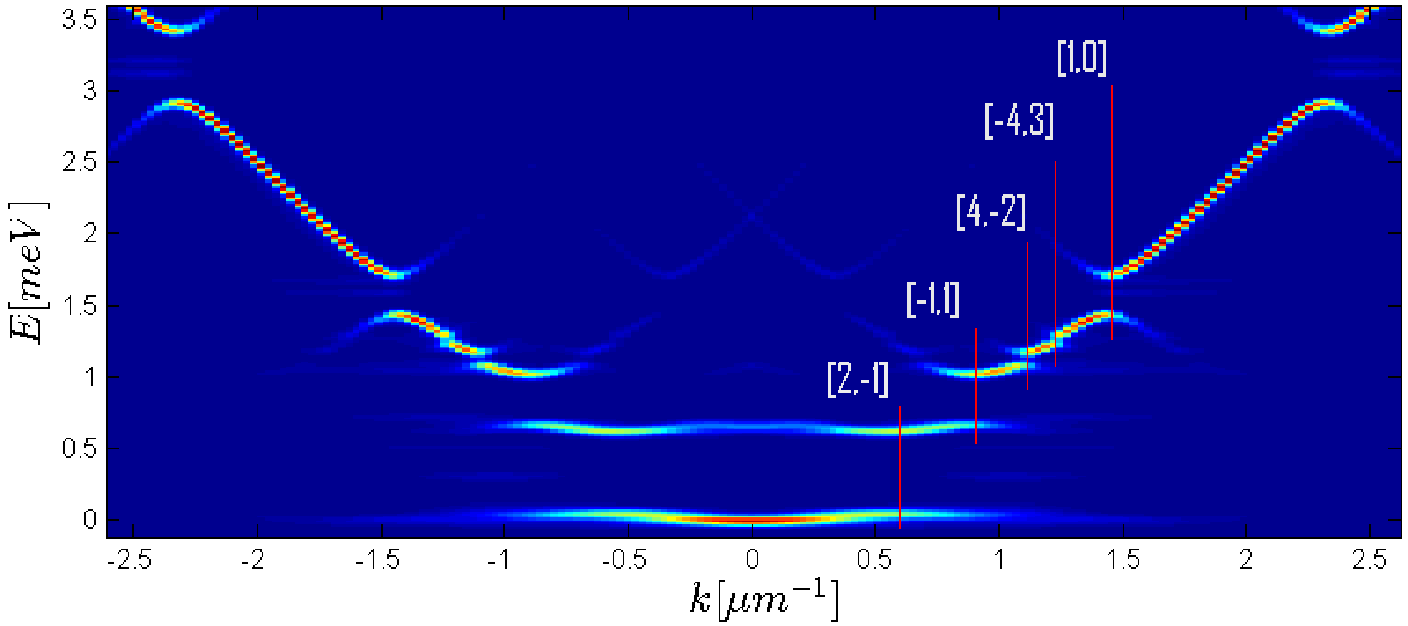


Calculating the integrated density of states (IDOS)

Integrated density of states (IDOS)-Gap labeling

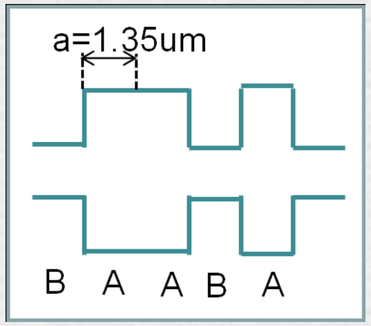
$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dx^2}+V(x)\right]\psi(x)=E\,\psi(x)$$

where

$$V(x)=u_b(x)\times\left[\chi(\sigma x)\sum_n \delta(x-na)\right]$$

Shape of each letter

$$\chi(x)=\begin{cases}1, & -\sigma<x<-\sigma^3\\ 0, & -\sigma^3<x<\sigma^2\end{cases}$$



$\sigma=\left(\sqrt{5}-1\right)/2$ is the inverse golden mean

$$V(k) = u_b(k) \times \sum_{p,q} \chi_q \, \delta\left(ka - 2\pi[p + \sigma q]\right)$$

Each pair $\{p,q\}$ of integers defines a unique Bragg peak (σ is irrational).

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$$\sigma \approx F_j / F_{j+1}$$

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Bragg peaks at values $k=Q\equiv\frac{1}{a}\left(F_{j+1}\,p+F_j\,q\right)\xrightarrow[j\to\infty]{}\frac{1}{a}(p+q\sigma)$

Perturbation theory (small V)

For the (quasi) crystal, a series of gaps open at each value of the (independent) Bragg peaks (Bloch thm.).

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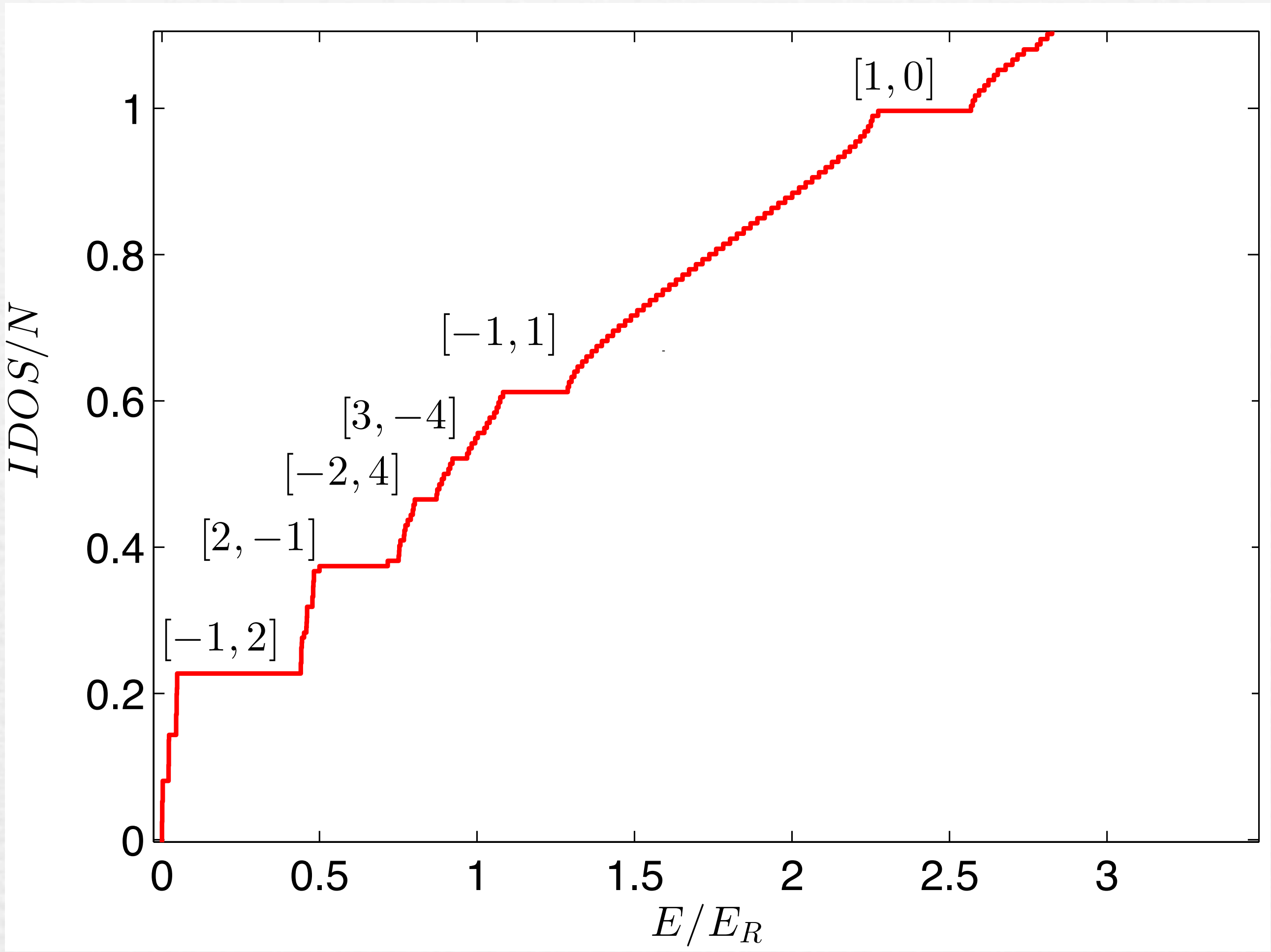
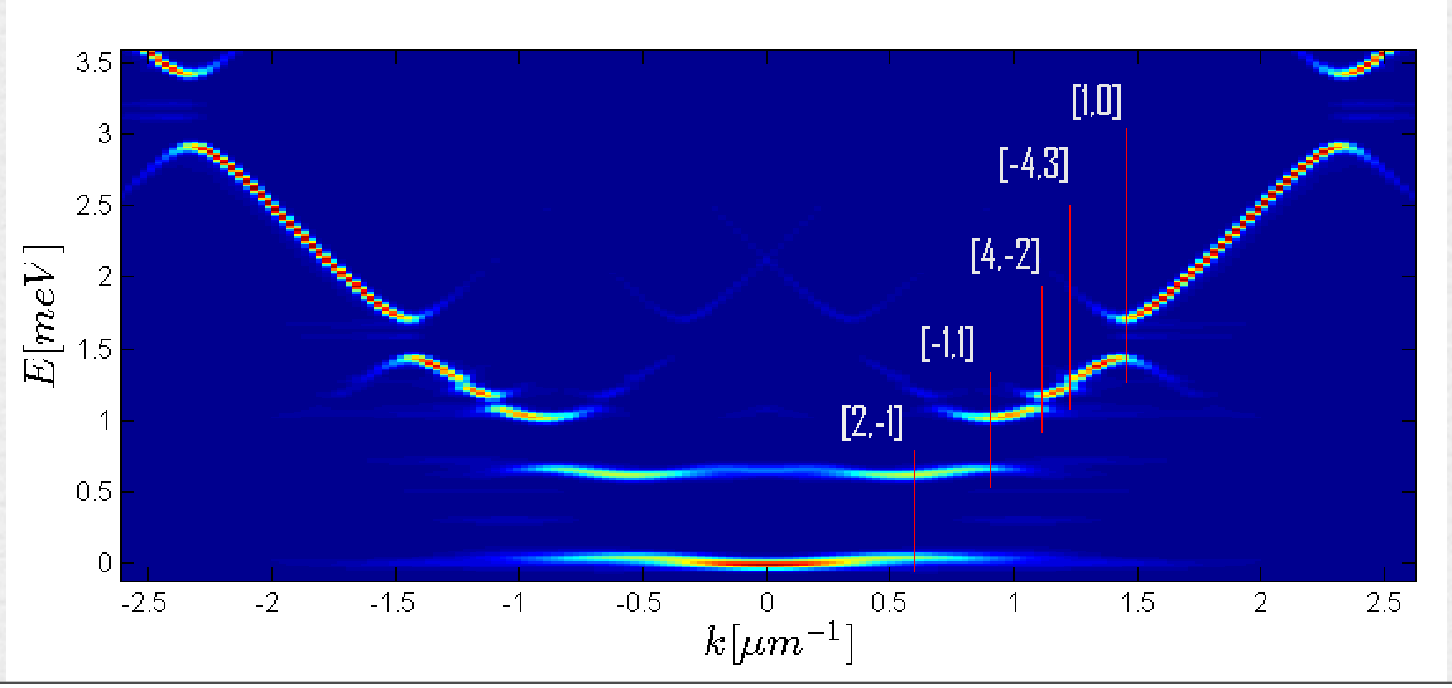
The (normalized) IDOS inside a gap labeled by $\{p,q\}$ is

$$N(E) = p + q\,\sigma \quad (\text{mod.}1)$$

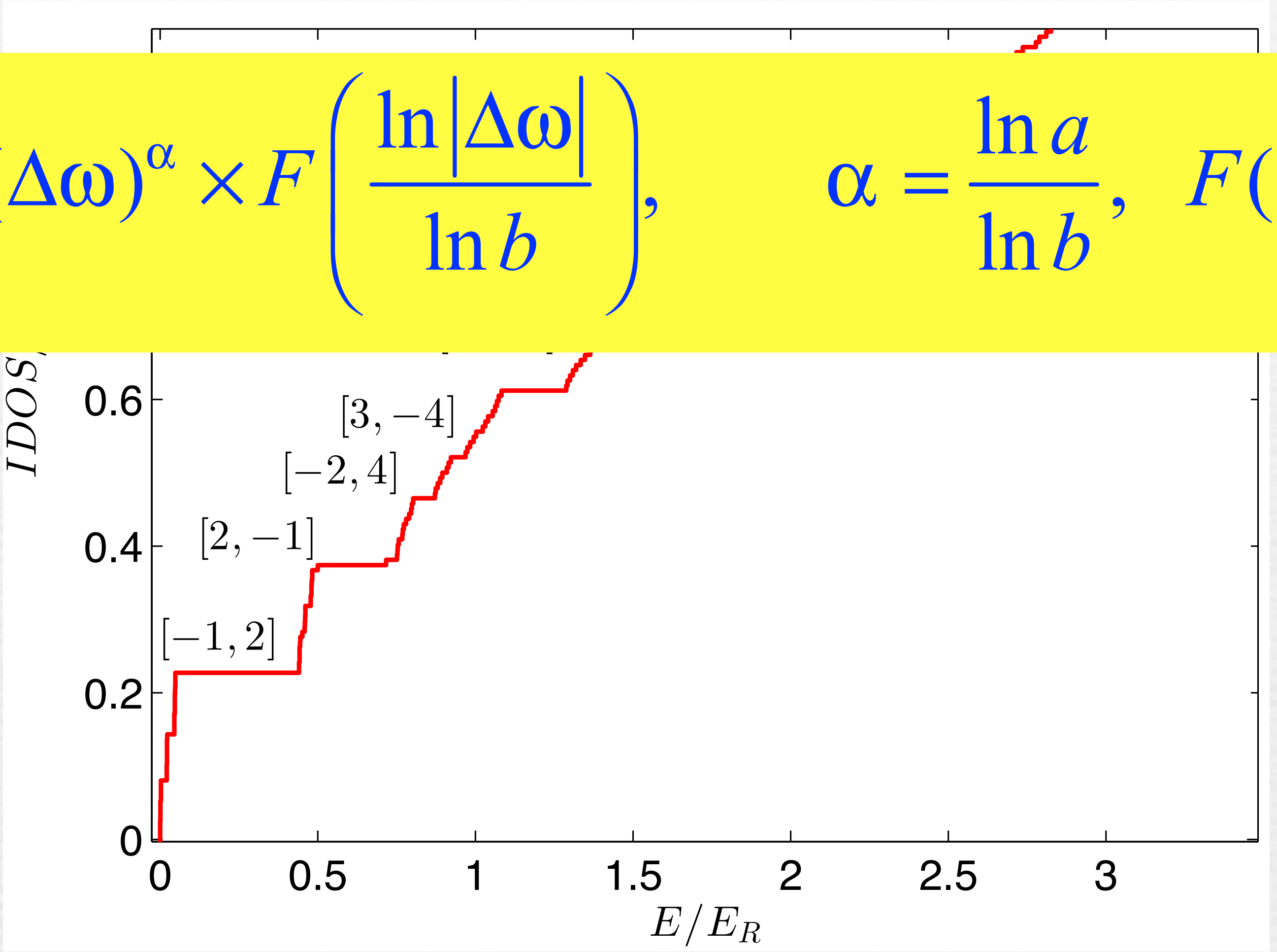
Integrated Density of States-Gap Labeling

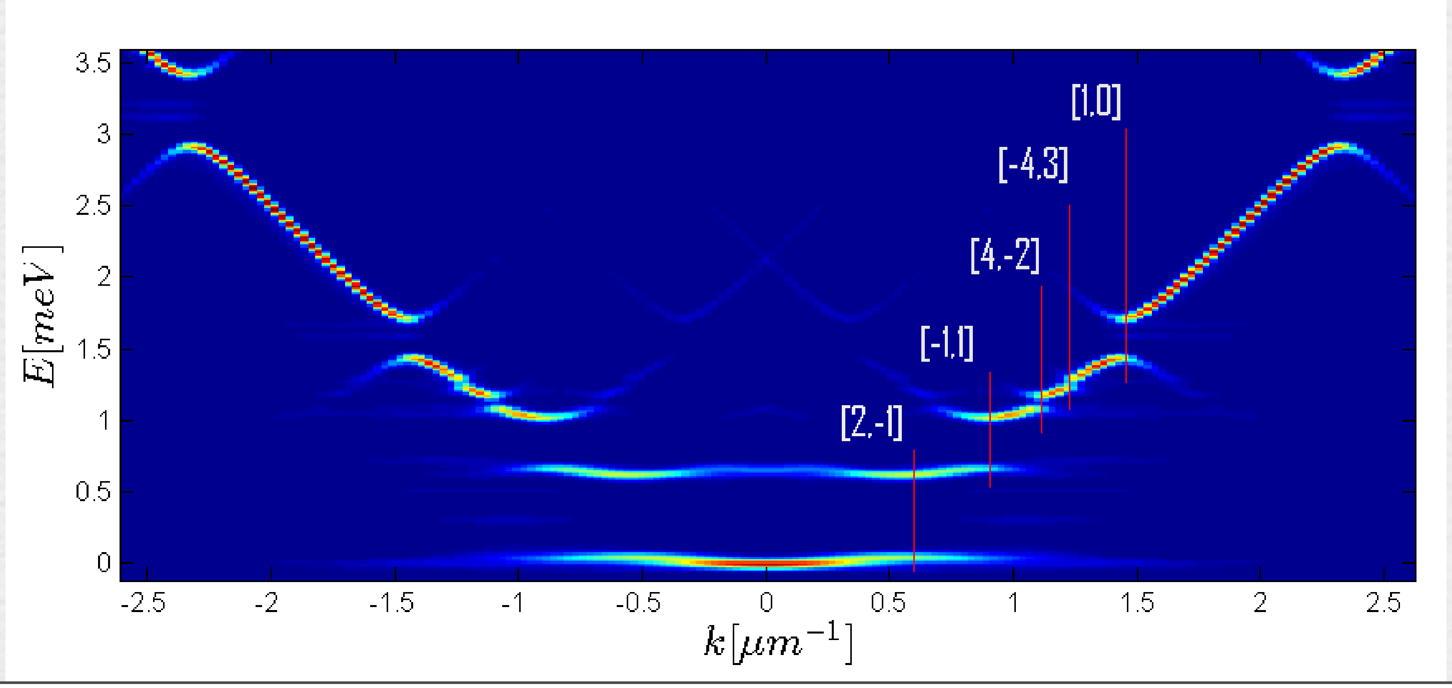




Integrated Density of States-Gap Labeling

$$N_\omega(\Delta\omega) = (\Delta\omega)^\alpha \times F\left(\frac{\ln|\Delta\omega|}{\ln b}\right), \qquad \alpha = \frac{\ln a}{\ln b}, \quad F(x+1) = F(x)$$





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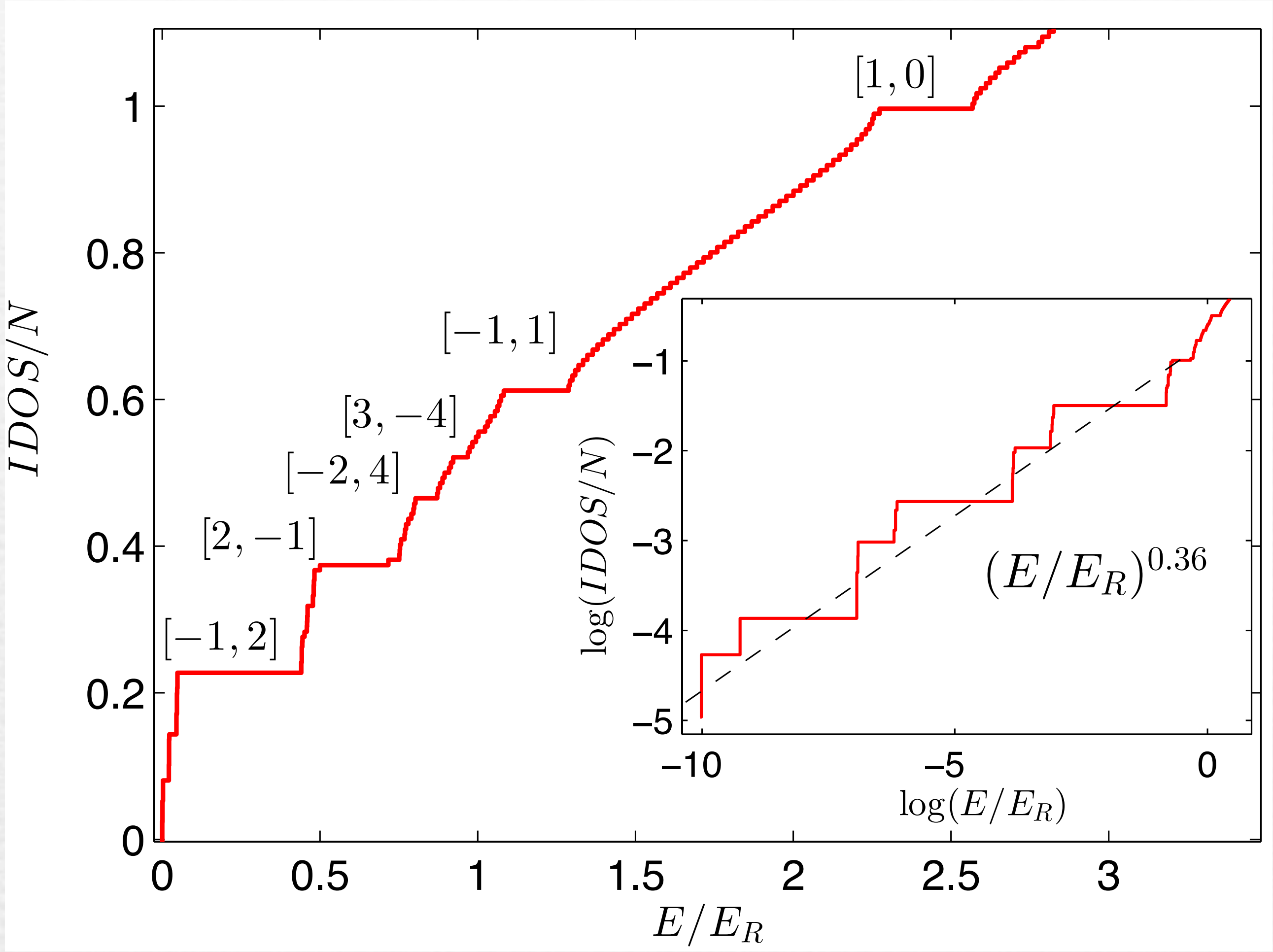
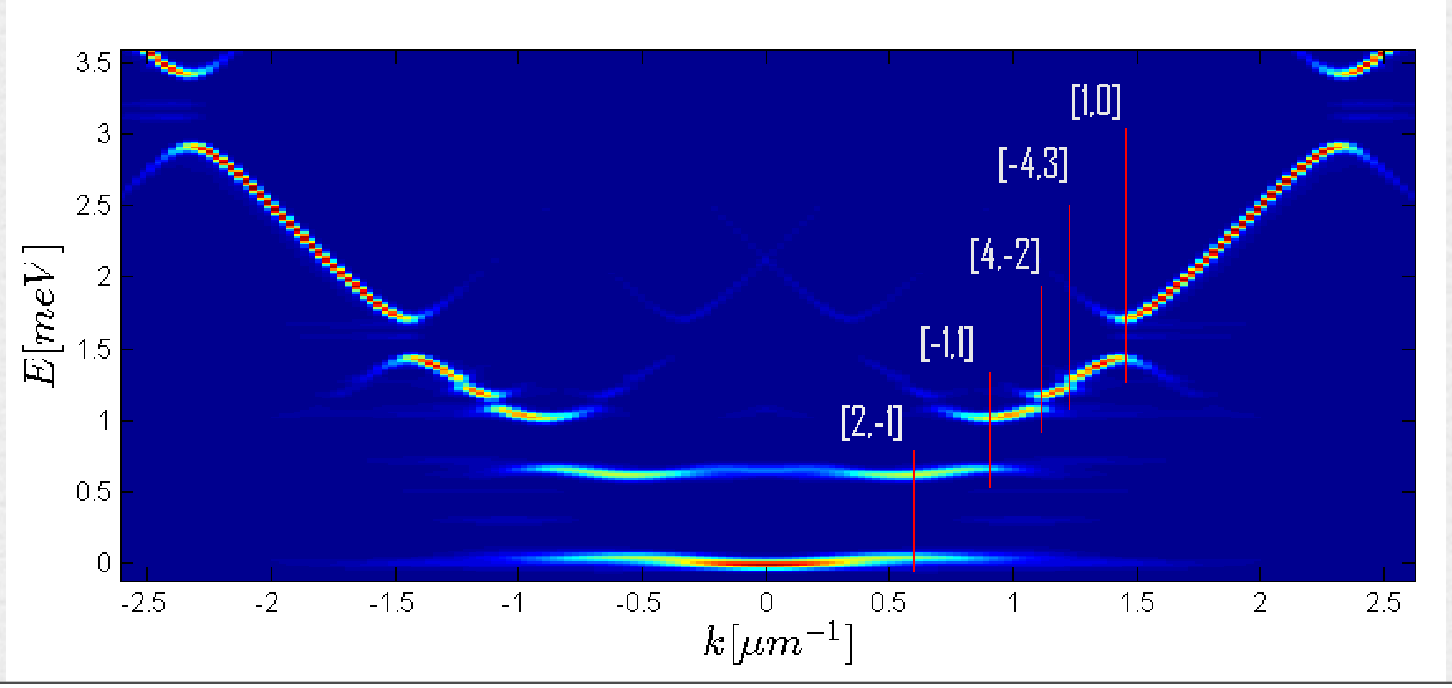
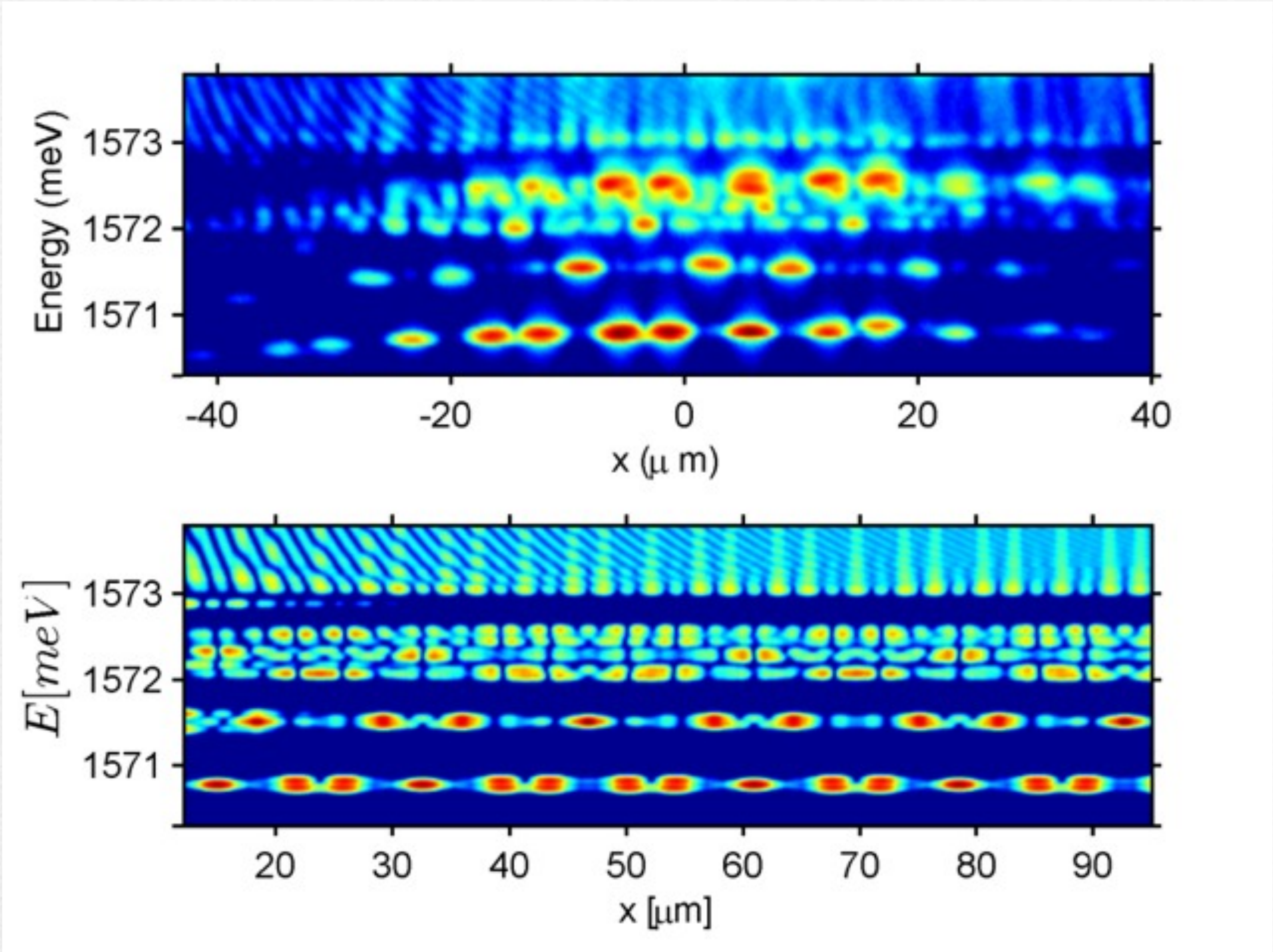




Spatial distribution - Localization of modes



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- An exactly solvable toy model suggests that a similar scaling takes place also for long times.
- The experimental study of a macroscopic coherent polariton gas in a Fibonacci cavity allows for a quantitative study of a fractal singular continuous energy spectrum : spectral function, wave functions and gap labeling.

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Further directions

- Generalization to other quantum fields : BEC, superfluidity and Off diagonal long range order (ODLRO) for massive bosons.