Superconducting hybrid systems and Majorana fermions

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Outline

I. Superconducting grains

1. Bardeen-Cooper-Schrieffer theory (1957) $\sigma = \left\{ \begin{matrix} \uparrow \text{ or } + \\ \downarrow \text{ or } - \end{matrix} \right.$

- $$H_{BCS} = \int dr \left\{ \sum_\sigma \psi^+_\sigma(r) \left(-\frac{\nabla^2}{2m}\right) \psi_\sigma(r) - |\lambda| V\, n_\uparrow(r)\, n_\downarrow(r) \right\}$$

free fermions + local attraction

$$\psi_\sigma(r) = \frac{1}{\sqrt{V}} \sum_k a_{k\sigma} e^{ikr} \qquad \varepsilon_k = \frac{k^2}{2m}$$

$$H_{BCS} = \sum_{k\sigma} \varepsilon_k a^+_{k\sigma} a_{k\sigma} - |\lambda| \sum_{kk'q} a^+_{k+q\uparrow} a^+_{k'-q\downarrow} a_{k'\downarrow} a_{k\uparrow}$$

- mean field in grand canonical ensemble:

$$a_{k'\downarrow} a_{k\uparrow} = \langle a_{-k\downarrow} a_{k\uparrow} \rangle \delta_{kk'} + \underbrace{a_{k'\downarrow} a_{k\uparrow} - \langle a_{-k\downarrow} a_{k\uparrow} \rangle \delta_{k'k}}_{\text{assumed to be small}}$$

$$H_{BCS} - \mu N \approx \sum_{k\sigma} \xi_k a^+_{k\sigma} a_{k\sigma} + \Delta \sum_k a^+_{k\uparrow} a^+_{-k\downarrow} + \Delta^* \sum_k a_{-k\downarrow} a_{k\uparrow} + \text{cste}$$

with $\Delta = -|\lambda| \sum_k \langle a_{-k\downarrow} a_{k\uparrow} \rangle$ $\qquad \xi_k = \varepsilon_k - \mu$

$\rightarrow$ quadratic form

- Bogoliubov transformation:

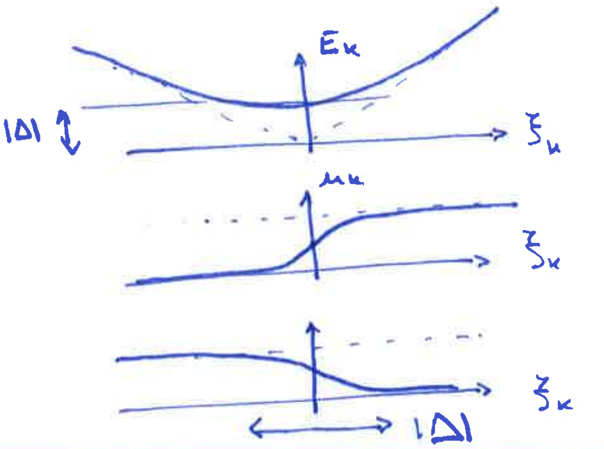
$$H_{BCS} - \mu N = \sum_k \left(a^+_{k\uparrow} \;\; a_{-k\downarrow} \right) \begin{pmatrix} \xi_k & \Delta \\ \Delta^* & -\xi_k \end{pmatrix} \begin{pmatrix} a_{k\uparrow} \\ a^+_{-k\downarrow} \end{pmatrix} + \text{cste}$$

$$= \sum_{k\sigma} E_k \alpha^+_{k\sigma} \alpha_{k\sigma} + \text{cste}$$

with $E_k = \sqrt{|\Delta|^2 + \xi_k^2}$

$$a_{k\sigma} = e^{i\frac{\varphi}{2}} \left(u_k \alpha_{k\sigma} - \sigma v_k \alpha^+_{-k-\sigma} \right)$$

$$u_k = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{\xi_k}{E_k}} \qquad v_k = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{\xi_k}{E_k}}$$



$\xi_k \gg \Delta$ $\qquad \alpha_{k\sigma} \approx a_{k\sigma}$ "particle"

$\xi_k \ll -\Delta$ $\qquad \alpha_{k\sigma} \approx -\sigma a_{-k-\sigma}$ "hole"

- ground state $\quad |BCS_\varphi\rangle = \prod_k \left(u_k - e^{i\varphi} v_k \, a^+_{k\uparrow} a^+_{-k\downarrow}\right)|\emptyset\rangle$

  such that $\quad \alpha_{k\sigma} |BCS_\varphi\rangle = 0$

  * $\Delta = 0$ (Normal state) $\quad |BCS_\varphi\rangle \propto \prod_{|k|<k_F,\sigma} a^+_{k\sigma} |\emptyset\rangle \quad$ is the Fermi sea

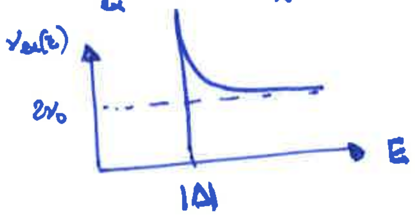
  * $\Delta \neq 0$ (Superconduct. state) $\quad$ correlated pairs with opposite spins & momenta
  Cooper pairs condense in the ground state

  $\rightarrow$ superconductivity destroyed by orbital or Zeeman effect of a magnetic field

NB: U(1) symmetry breaking $\leftrightarrow$ Δ defined up to a global phase

- excitation spectrum is gapped:

per spin: $\quad \nu_{ex}(E) = \sum_k \delta(E - E_k) = 2\nu_0 \dfrac{E}{\sqrt{E^2 - |\Delta|^2}} \, \theta(E - |\Delta|)$



$\nu_0 = \dfrac{m k_F}{2\pi^2}$ density of states per spin in N state

$\rightarrow$ Gap 2Δ by breaking a Cooper pair into two excitations (optics, SIS junctions)
$\quad$ Gap Δ by injecting/extracting electrons (SIN junction)

NB: excitations are a mixture of particles & holes

NB: arbitrariness in labelling states due to a built-in electron-hole symmetry

quasiparticle representation $\qquad$ or $\qquad$ semiconductor representation

using $\gamma^+_\uparrow(E) = \gamma_\downarrow(-E)$

2 - finite-size grains

- projection onto canonical ensemble

$$|BCS_N\rangle = \int_0^{2\pi} \frac{d\varphi}{2\pi}\, e^{-i\frac{N\varphi}{2}}\, |BCS_\varphi\rangle$$ for an even number N of electrons

and similarly for excited states

- $T \ll \Delta$

even state: all electrons form Cooper pairs $\quad Z_e \approx 1$

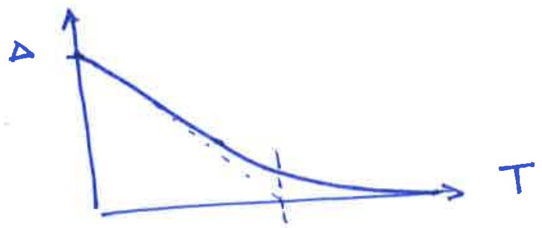
odd state: 1 unpaired electron $\quad Z_o \approx \sum_{k\sigma} e^{-\frac{E_{k\sigma}}{T}}$

$$Z_o \approx 2\underbrace{\nu_0 V}_{\frac{1}{\delta}}\int d\xi\, e^{-\frac{\sqrt{\Delta^2+\xi^2}}{T}} \approx \frac{2}{\delta}\int d\xi\, e^{-\frac{\Delta}{T}} e^{-\frac{\xi^2}{2\Delta T}} \approx \frac{\sqrt{8\pi T\Delta}}{\delta}$$

δ: mean level spacing

$$F_{e/o} = -T\ln Z_{e/o}$$

$$F_o - F_e = \Delta - T\ln\frac{\sqrt{8\pi T\Delta}}{\delta}$$



$$T^* \approx \frac{\Delta}{\ln\frac{\Delta}{\delta}}$$

→ the ratio $\frac{\Delta}{\delta}$ sets the criterion for smallness of fluctuations and validity of mean field treatment

$$\frac{\Delta}{\delta} \sim \Delta m k_F L^3 \sim \frac{L^3}{\lambda_F^2 \xi_s} \gg 1$$

where $\xi_s \sim \frac{v_F}{\Delta}$ typical "size" of Cooper pairs.

i.e. superconductivity persists in small grains of size $L \sim (\xi_s \lambda_F^2)^{1/3} \ll \xi_s^3$

[Anderson, 1959]

→ The superconducting properties of grains with even or odd number of electrons cannot be distinguished above a temperature $T^* \sim \frac{\Delta}{\ln \Delta/\delta}$ much smaller than the critical temperature $T_c \sim \Delta$.

• Coulomb staircase

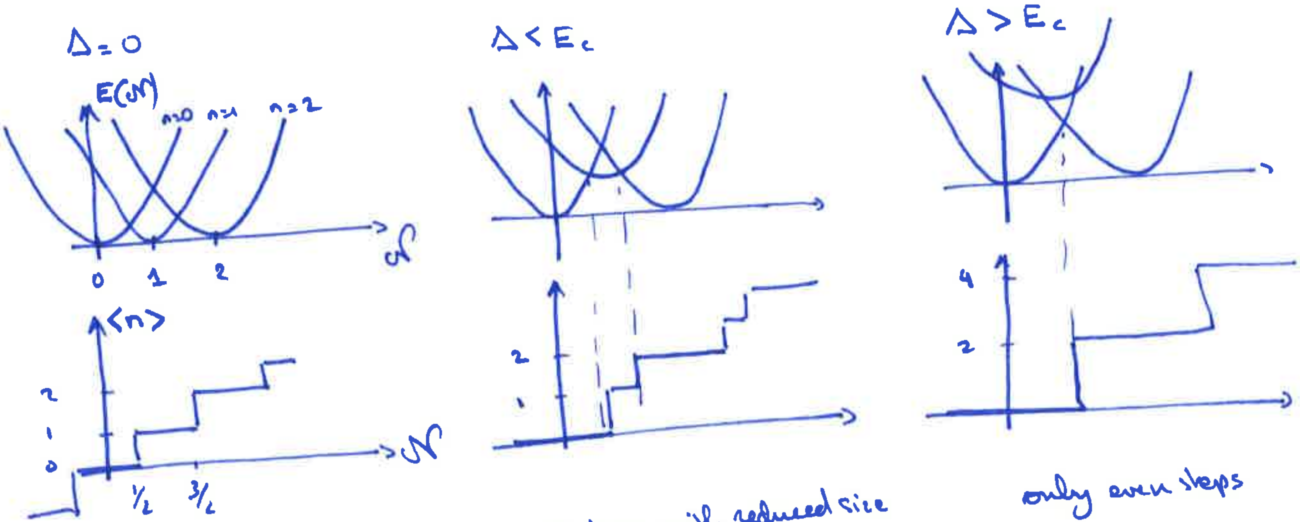
$$E_{el}(q) = \frac{1}{2C_\Sigma}(q + C_g V_g)^2 + cste$$

$$q = -en$$

$$E_n(\mathcal{N}) = E_c(n - \mathcal{N})^2 + \frac{1-(-1)^n}{2}\Delta$$

at $T \ll T^*$

$$\mathcal{N} = \frac{C_g V_g}{e}$$



odd steps with reduced size

only even steps

NB: requested conditions

$T \ll T^*, E_c$

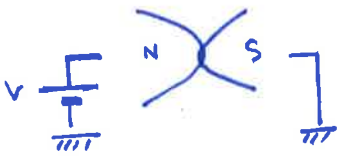
$R \gg R_Q = \frac{h}{e^2}$ $\longleftrightarrow$ $Z_{AC} \gg \frac{\hbar}{E_c}$

II - Tunnel junctions

$(e>0)$

1 - N/I/S junction

I : insulating region



$$H = H_L + H_R + H_T$$

$$= \sum_{k\sigma} (\xi_k - eV) a^+_{k\sigma} a_{k\sigma} + \sum_{p\sigma} E_p \beta^+_{p\sigma} \beta_{p\sigma} + \sum_{kp\sigma} t_{kp\sigma} a^+_{k\sigma} \underbrace{b_{p\sigma}}_{e^{\frac{i\varphi}{2}}[u_p \beta_{p\sigma} - \sigma v_p \beta^+_{-p-\sigma}]} + h.c.$$

for simplicity, we consider $t_{kp\sigma} \simeq t_0$ corresponding to a point contact

a - quasiparticle current

- Fermi Goldenrule applied with H_T

$$\Gamma_{R\to L} = 2\pi |t_0|^2 \sum_{kp\sigma} \left[u_p^2 f_p (1-f_k)\, \delta(\xi_k - eV - E_p) + v_p^2 (1-f_p)(1-f_k)\, \delta(\xi_k - eV + E_p) \right]$$

$$\Gamma_{L\to R} = 2\pi |t_0|^2 \sum_{kp\sigma} \left[u_p^2 (1-f_p) f_k\, \delta(\xi_k - eV - E_p) + v_p^2 f_p f_k\, \delta(\xi_k - eV + E_p) \right]$$

$$I = e(\Gamma_{R\to L} - \Gamma_{L\to R})$$

$$= 4\pi e |t_0|^2 \underbrace{\sum_{kp}}_{\int_0^{\infty} dE\, \delta(E-E_p) \int_{-\infty}^{+\infty} d\xi\, \delta(\xi - \xi_k)} \left[\underbrace{u_p^2 (f_p - f_k)\, \delta(\xi_k - eV - E_p) + v_p^2}_{\hookrightarrow \text{ only } \frac{1}{2} \text{ survives the sum}} (1 - f_p - f_k)\, \delta(\xi_k - eV + E_p) \right]$$

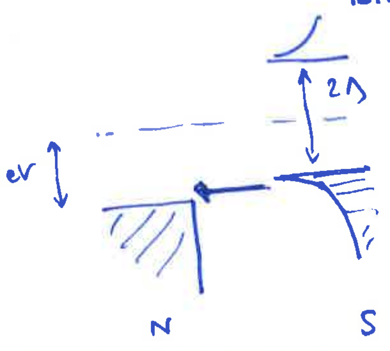
$$= 2\pi e |t_0|^2 \int_0^{\infty} dE \int_{-\infty}^{+\infty} d\xi\, \nu_{s}(E)\, \nu_N(\xi) \left\{ [f(E) - f(\xi)]\, \delta(\xi - E - eV) + [1 - f(E) - f(\xi)]\, \delta(\xi + E - eV) \right\}$$

$$= 4\pi e |t_0|^2 \int_{-\infty}^{\infty} dE \int_{-\infty}^{+\infty} d\xi\, \nu_S(E)\, \nu_N(\xi) [f(E) - f(\xi)]\, \delta(\xi - E - eV)$$

where $\nu_S(E) = \nu_0 \dfrac{|E|}{\sqrt{E^2 - |\Delta|^2}}\, \theta(|E| - |\Delta|)$

convolution of DOS ponderated with Fermi functions

- $$I = \frac{G_N}{e} \int_{|E|>\Delta} dE\, \frac{|E|}{\sqrt{E^2 - \Delta^2}} \left[f(E) - f(E + eV) \right]$$ $\qquad G_N = 4\pi e^2 \nu_L \nu_R |t_0|^2$



N state $*$ $I = G_N V$

S state $*$ $I = \dfrac{G_N}{e} \sqrt{(eV)^2 - \Delta^2}$ at $\begin{array}{l} eV > \Delta \\ T = 0 \end{array}$

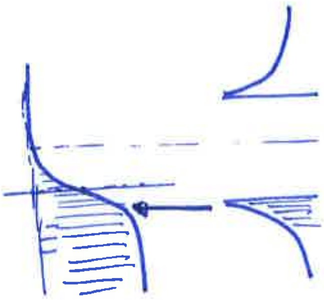
$$\frac{\partial I}{\partial V} = G_N \frac{eV}{\sqrt{(eV)^2 - \Delta^2}}$$

$*$ $\left.\dfrac{\partial I}{\partial V}\right|_{V\to 0} \propto e^{-\Delta/T}$

applications: * tunneling spectroscopy (measure DOS)

* thermometry

* cooling:



$T < \Delta - eV < \Delta$

$\dot{Q}_{\text{out of } N} > 0$

• "Landauer" formulation:

$G_N = \frac{2e^2}{h} D$ ← transmission $\qquad D = 4\pi^2 \nu_L \nu_R |t_0|^2 \ll 1$

in NIN junction $\qquad I = G_N V \qquad$ remains true up to $D = 1$

(higher order in t_0: $\quad D = \frac{4\alpha}{(1+\alpha)^2} \qquad \alpha = \pi^2 \nu_L \nu_R |t_0|^2$)

b_ Andreev current

• $T, eV \ll \Delta$: negligible quasiparticle current

But Bogoliubov quasiparticles may be excited virtually

↳ 2^{nd} order perturbation theory for quasiparticles with energy $\ll \Delta$

$$H_{eff} = H_N - H_T \frac{1}{H_S} H_T$$

with $H_T = t_0 \sum_{kp\sigma} a^+_{k\sigma} \left[u_p \beta_{p\sigma} - \sigma v_p \beta^+_{-p-\sigma} \right] + h.c.$

$= t_0 \sum_{kp\sigma} \beta^+_{p\sigma} \left[u_p a_{k\sigma} - \sigma v_p a^+_{-k-\sigma} \right] + h.c.$

$$\hookrightarrow H_{eff} = H_N - t_0^2 \sum_{kk'p\sigma} \left[u_p a^+_{k'\sigma} - \sigma v_p a_{-k'-\sigma} \right] \frac{1}{E_p} \left[u_p a_{k\sigma} - \sigma v_p a^+_{-k-\sigma} \right]$$

$$= H_N - t_0^2 \sum_{kk'\sigma} \left\{ \underbrace{\left(\sum_p \frac{u_p^2 - v_p^2}{\varepsilon_p} \right)}_{0} a^+_{k'\sigma} a_{k\sigma} - \sigma \underbrace{\left(\sum_p \frac{u_p v_p}{\varepsilon_p} \right)}_{\frac{\pi \nu_{S0}}{2}} a^+_{k\sigma} a^+_{k'-\sigma} + h.c. \right\}$$

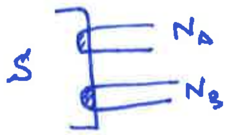
no additional potential scattering

applications: * point contact at a perfect barrier between S and a ferromagnet

$$G_A = \frac{4e^2}{h} N_{min}$$

$$G_N = \frac{e^2}{h}\left(N_{min} + N_{maj}\right)$$

$\rightarrow$ access to the spin polarization $\underline{P} = \dfrac{N_{maj} - N_{min}}{N_{maj} + N_{min}}$

* 

$$H_{eff} = H_A + H_B + \sum_{\substack{kk' \\ \alpha\beta = A,B}} t_{\alpha\beta}\, a^{+}_{k\alpha\uparrow} a^{+}_{k'\beta\downarrow} + h.c.$$

$$+ \sum_{kk'\sigma} t'_{AB}\, a^{+}_{kA\sigma} a_{k'B\sigma} + h.c.$$

$$t_{\alpha\beta} \sim \cos^2(k_F x_{\alpha\beta})\, e^{-\frac{|x_{\alpha\beta}|}{\xi_S}}$$ crossed Andreev reflection

$$t'_{AB} \sim \sin^2(k_F x_{AB})\, e^{-\frac{|x_{AB}|}{\xi_S}}$$ elastic cotunneling

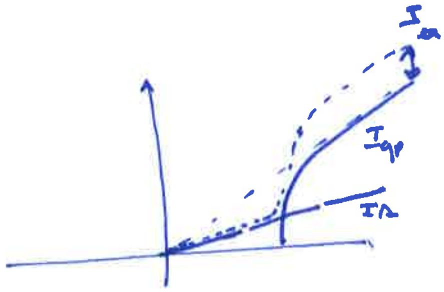
CAR: production of entangled pairs of electrons !?

$$H_{eff} = H_N + \pi \nu_{S0} t_0^2 \sum_{kk'} \underbrace{a^+_{k\uparrow} a^+_{k'\downarrow}}_{\text{creation of a singlet \underline{Andreev pair} in N}} + h.c.$$

- $$\Gamma_{R\to L} = 2\pi \left(\pi \nu_{S0} t_0^2\right)^2 \sum_{kk'} (1-f_k)(1-f_{k'})\, \delta(\xi_k + \xi_{k'} - 2eV)$$

$$\Gamma_{L\to R} = \text{———} \; f_k f_{k'} \; \text{———}$$

$$I_A = (2e)\left(\Gamma_{R\to L} - \Gamma_{L\to R}\right)$$
$$= 2e \cdot 2\pi \left(\pi \nu_{S0} t_0^2\right)^2 \cdot \nu_{N_0}^2 \, 2eV$$
$$= G_A V$$
$$G_A = \frac{4e^2}{h}\left(\frac{D}{2}\right)^2$$



$I \sim I_A + I_{qp}$

↳ subgap current

↳ excess current $\quad I - G_N V \big|_{V \gg \Delta/e} \neq 0$

↳ higher order in D: $\quad G_A\big|_{V\to 0} = \frac{4e^2}{h} R_A \qquad R_A = \left(\frac{D}{2-D}\right)^2$

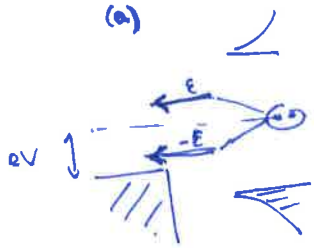
- Andreev reflection:

(a) 1 Cooper pair ⟶ two electrons with opposite energies (with respect to chemical potential in S)

or equivalently:

(b) 1 electron is reflected into a hole at the same energy while a Cooper pair is created in S

(a)



works at $\propto \varepsilon < eV$

↳ slopes $-eV$ for electron transfer

(b)

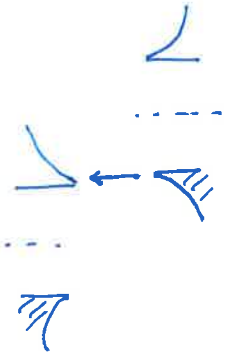
slope $-eV$ for electron
$+eV$ for holes

2_ S/I/S junctions

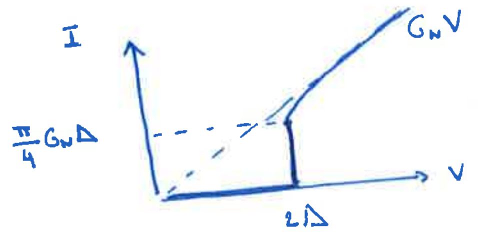
a_ quasiparticle current

• tunneling regime

$$I(V) = \frac{G_N}{e} \int_{\substack{|E|>\Delta \\ |E+eV|>\Delta}} dE \, \frac{|E|}{\sqrt{E^2-|\Delta|^2}} \, \frac{|E+eV|}{\sqrt{(E+eV)^2-|\Delta|^2}} \left[f(E) - f(E+eV) \right]$$



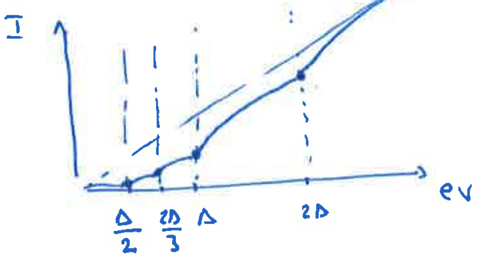
T = 0 threshold at $eV = 2\Delta$



• higher orders = multiple Andreev reflections
(= multiparticle tunneling)

new conduction channels open when $eV_n = \frac{2\Delta}{n}$ (n integer)

subgap resonances:



b_ Andreev bound states

• In equilibrium, there are no propagating states below the superconducting gap. However, there may be localized states as we discuss in two limits:

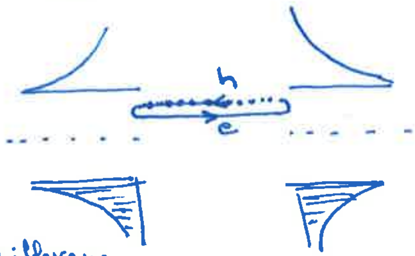
ballistic case:

according to the Bogoliubov transformation, the particle and of an evanescent state in the leads

hole components satisfy $\dfrac{v(E)}{u(E)} = \sqrt{\dfrac{E-\xi}{E+\xi}}$ with $\xi = \sqrt{E^2-|\Delta|^2}$ $= \pm i\sqrt{|\Delta|^2-E^2}$ at $|E| < |\Delta|$

i.e. $\dfrac{v(E)}{u(E)} = e^{\mp i \arccos\left(\frac{E}{\Delta}\right)}$

resonance condition:



$\pm 2\arccos\dfrac{E}{\Delta} + \varphi = 2n\pi$

↑

superconducting phase difference

i.e. $E_A = \pm \Delta \cos\dfrac{\varphi}{2}$

tunneling regime $\qquad\qquad \varphi = \varphi_L - \varphi_R$

$$H = \sum_{k\sigma} E_k \alpha^+_{k\sigma}\alpha_{k\sigma} + \sum_{p\sigma} E_p \beta^+_{p\sigma}\beta_{p\sigma}$$
$$+ t_0 \sum_{kp\sigma} e^{-i\varphi}\left[u_k \alpha^+_{k\sigma} - \sigma v_k \alpha_{-k-\sigma}\right]\left[u_p \beta_{p\sigma} - \sigma v_p \beta^+_{-p-\sigma}\right] + h.c.$$

$$\xi_k \to 0 \qquad E_k = \Delta + \frac{\xi_k^2}{2\Delta} \qquad u_k, v_k \approx \frac{1}{\sqrt{2}}$$

we project H on the subspace of states with energy close to the continuum threshold Δ and we assume that an excited state contains at most one excitation

$$\hookrightarrow H_{eff} = \sum_{k\sigma}\left(\Delta + \frac{\xi_k^2}{2\Delta}\right)\alpha^+_{k\sigma}\alpha_{k\sigma} + \sum_{p\sigma}\left(\Delta + \frac{\xi_p^2}{2\Delta}\right)\beta^+_{p\sigma}\beta_{p\sigma}$$
$$+ i t_0 \sin\varphi \sum_{kp\sigma}\left(\beta^+_{p\sigma}\alpha_{k\sigma} - \alpha^+_{k\sigma}\beta_{p\sigma}\right)$$

we introduce $\quad \gamma_{k\sigma} = \dfrac{1}{\sqrt{2}}\left(\alpha_{k\sigma} + i\beta_{k\sigma}\right)$

$\qquad\qquad\qquad \delta_{k\sigma} = \dfrac{1}{\sqrt{2}}\left(\alpha_{k\sigma} - i\beta_{k\sigma}\right)$

$$H_{eff} = \sum_{k\sigma}\left(\Delta + \frac{\xi_k^2}{2\Delta}\right)\left(\gamma^+_{k\sigma}\gamma_{k\sigma} + \delta^+_{k\sigma}\delta_{k\sigma}\right)$$
$$+ t_0 \sin\varphi \sum_{kk'\sigma}\left(-\gamma^+_{k\sigma}\gamma_{k'\sigma} + \delta^+_{k\sigma}\delta_{k'\sigma}\right)$$

This Hamiltonian is identical to that of a massive particle in the presence of a local potential barrier or well

$$H_{eff} = \Delta + \frac{p^2}{2m} \pm U_0\,\delta(x) \qquad (U_0 > 0)$$

which admits a bound state in the case of the well.

to obtain the bound state energy, we look for a trial wavefunction $|\psi\rangle = \sum_k C_k \gamma^+_{k\uparrow} |\phi\rangle$

$$\Rightarrow \left(E - \Delta - \frac{\xi_k^2}{2\Delta}\right) C_k + \sum_q t \sin\frac{\varphi}{2}\, C_q = 0$$

$$1 = t_0 \sin\frac{\varphi}{2} \sum_k \frac{1}{\Delta - E + \frac{\xi_k^2}{2\Delta}}$$

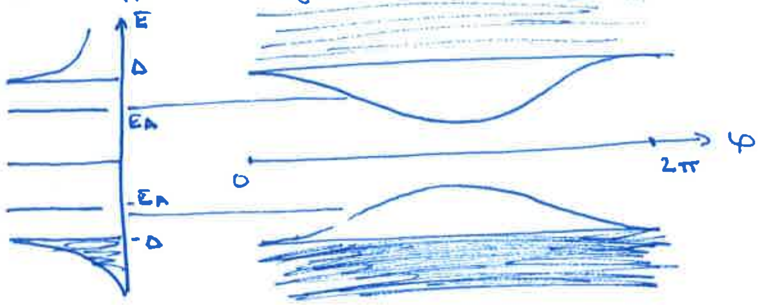
admet une solution $E < \Delta$ si $0 < \varphi < \pi$ $(t_0 > 0)$

$$1 = \pi \nu_s t \sin\frac{\varphi}{2} \sqrt{\frac{2\Delta}{\Delta - E}}$$

i.e. $E_A = \Delta\left(1 - \frac{D}{2}\sin^2\frac{\varphi}{2}\right)$

arbitrary transparency : $E_A = \Delta\sqrt{1 - D\sin^2\frac{\varphi}{2}}$

- In the semiconductor representation, we obtain two ABS with opposite energies



In equilibrium, the energy of the junction depends on the phase difference : $E_J(\varphi) = -E_A(\varphi) f(-E_A(\varphi)) + E_A(\varphi) f(E_A(\varphi))$
+ contribution of the continuum

$$= -E_A(\varphi) \operatorname{th}\frac{E_A(\varphi)}{2T} + \text{cste}.$$

NB: tunneling case : $E_J(\varphi) \approx -E_J \operatorname{th}\frac{\Delta}{2T}\cos\varphi + \text{cste}$

with $E_J = \frac{\pi}{4e^2} G_N \Delta$

NB : electromagnetic gauge:
if we perform a time-dependent $\underbrace{\text{transformation}}$ with $U = e^{-i\frac{\varphi(t)}{2}\sum_{k\sigma} a^+_{k\sigma} a_{k\sigma}}$ on H_{BCS}, we obtain an equivalent Hamiltonian

$$H_{eq} = U^+ H_{BCS} U - i\hbar U^+ \dot{U}$$

$$= \sum_{k\sigma}\Big(\xi_k - \underbrace{\frac{\hbar\dot\varphi}{2}}_{eV}\Big) a^+_{k\sigma} a_{k\sigma} + \Delta e^{i\varphi(t)} \sum_k a^+_{k\uparrow} a^+_{-k\downarrow} + \text{h.c.}$$

i.e. an electric potential comes with a time-dependent phase, with
$\boxed{\dot\varphi = 2eV(t)/\hbar}$

Josephson effects:

• equilibrium

the free energy variation can be related with the electric work

$$dE = I \times V\, \delta t = I \times \frac{\hbar}{2e}\, \delta\varphi$$

Thus $$I = \frac{2e}{\hbar} \frac{\partial E}{\partial \varphi}$$

$$I = \frac{\pi}{2} \frac{G_N \Delta}{e} \frac{\sin\varphi}{\sqrt{1 - D\sin^2\frac{\varphi}{2}}}\, \mathrm{th}\left[\frac{\Delta}{2T}\sqrt{1 - D\sin^2\frac{\varphi}{2}}\right]$$

* the Josephson relation $I = I_c \sin\varphi$ with critical current $I_c = \frac{\pi}{2}\frac{G_N \Delta}{e}$ (Ambegaokar - Baratoff relation) is recovered at $T=0$ for a tunnel junction
* the Josephson relation may contain higher harmonics
* the current flowing through the junction depends on the occupation of the ABS

current I $-I$ 0 0

poisoned states differ by a charge $e\,[2e]$

• non equilibrium

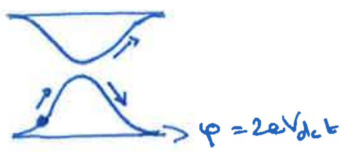
i) $I(t) = I_c \sin[\varphi(t)]$ without changing the occupation of ABS

→ $V(t) = V_{dc}$ Josephson radiation at frequency $\omega = 2eV_{dc}$

→ $V(t) = V_{dc} + V_{ac}\cos\Omega t$

current steps (Shapiro steps) at $2eV_{dc} = n\Omega$ (n integer)

ii)



a Landau - Zener process may switch the occupation of the ABS

Energy 2Δ is then transferred during period $T = \frac{2\pi}{2eV_{dc}}$ with probability $P_{LZ} = e^{-\frac{R\Delta}{eV}}$ $R \ll 1$

i.e. $$I(V) = \frac{2e\Delta}{\pi} e^{-\frac{R\Delta}{eV}}$$

→ There is also a connection between MAR and ABS

3. QD/S

spin degenerate

We describe a quantum dot that contains a discrete level with strong Coulomb repulsion (assuming a large level spacing in order to ignore the effect of other levels) $\longrightarrow$ Anderson model

$$H = H_S + \sum_\sigma \varepsilon d_\sigma^+ d_\sigma + U n_\uparrow n_\downarrow + H_T$$

$$n_\sigma = d_\sigma^+ d_\sigma$$

$$H_T = t \sum_{k\sigma} a^+_{k\sigma} d_\sigma + \text{c.c.}$$

Energy scales: ε, U, Δ, $\Gamma = 2\pi \nu_S |t|^2$

level broadening due to its coupling with the lead (in N state)

a. singlet / doublet transition

We assume $\Delta \gg \varepsilon, U, \Gamma$

then we can derive a low energy Hamiltonian in the 2^{nd} order in H_T:

$$H_{eff} = H_{dot} - H_T \frac{1}{H_S} H_T$$

$$= \varepsilon \sum_\sigma d_\sigma^+ d_\sigma + U n_\uparrow n_\downarrow + \Gamma d_\uparrow^+ d_\downarrow^+ + \text{h.c.}$$

eigenstates: $|\uparrow\rangle = d_\uparrow^+ |\emptyset\rangle$ with energy ε

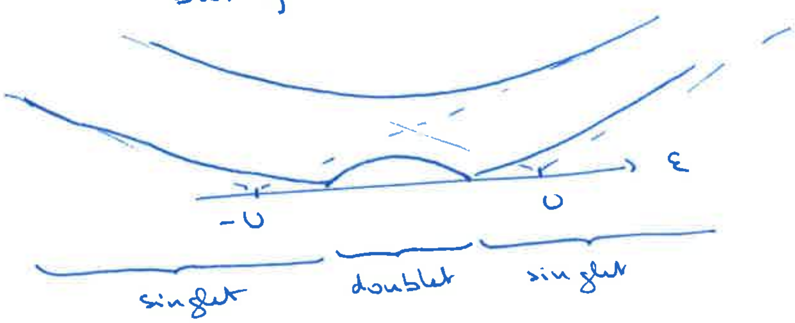
$|\downarrow\rangle = d_\downarrow^+ |\emptyset\rangle$

$|\pm\rangle = (\alpha_\pm + \beta_\pm d_\uparrow^+ d_\downarrow^+)|\emptyset\rangle$ with energy $E_\pm = \varepsilon + \frac{U}{2} \pm \sqrt{\left(\varepsilon+\frac{U}{2}\right)^2 + \Gamma^2}$

$|-\rangle$ and $|\sigma\rangle$ compete for being the ground state

$|-\rangle$ is always the ground state if $U < 2\Gamma$

but if $U > 2\Gamma$



b. competition between superconductivity and Kondo effect

- At large U, a Kondo effect arises in N state between the doublet state in the dot and the electrons in the lead $\longrightarrow$ formation of a Kondo singlet

  How does it resist to superconducting correlations?

  Kondo Hamiltonian (for ε close to $-U/2$) after projecting out the high energy states $|\phi\rangle$ and $d^+_\uparrow d^+_\downarrow|\phi\rangle$ of the dot

$$H = H_S + J \sum_{kk'\sigma\sigma'} a^+_{k\sigma}\, \vec{S}\cdot\vec{\sigma}_{\sigma\sigma'}\, a_{k'\sigma'}$$

$$\downarrow$$

J is related with a Kondo energy scale T_K and the bandwidth D of H_S through $J \sim \dfrac{1}{\ln \frac{D}{T_K}}$

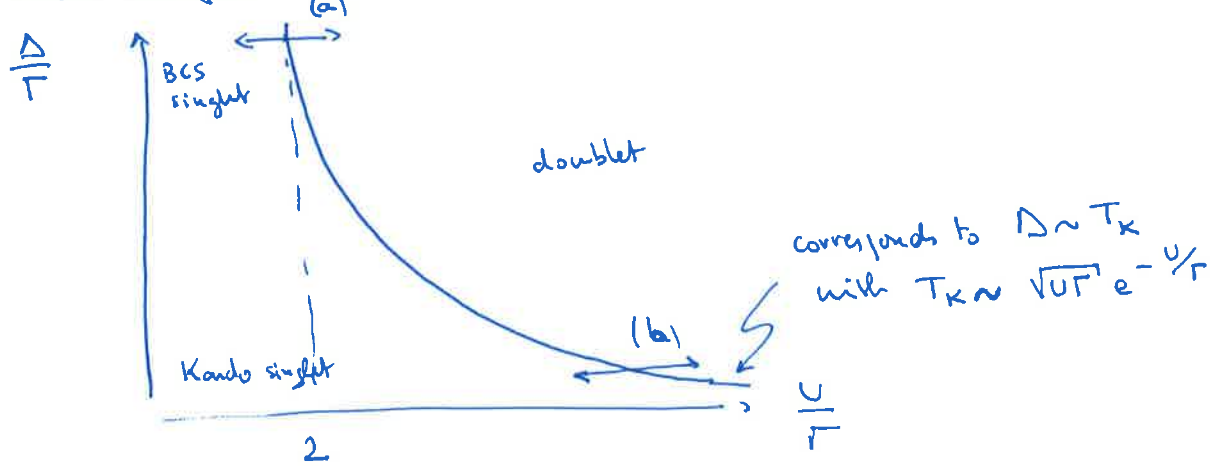
- At small J, we may project H on the space of states with energy close to Δ (like in II-2-a) and find an ABS with energy $E_A = \Delta(1 - J^2)$
$$\approx \Delta\left(1 - \frac{1}{\ln^2 \frac{\Delta}{T_K}}\right)$$

i.e. $\Delta \gg T_K$: doublet ground state
singlet excited state with energy E_A close to Δ

as $T_K \nearrow$, $E_A \searrow$ and eventually crosses the Fermi level

$\longrightarrow$ the parity of the ground state changes and it becomes a (Kondo) singlet

Phase diagram



III - Proximity effects

How do superconducting correlations leak into a non-superconducting metal?

1 - N/S hybrids

a - ballistic systems

- In long, ballistic 1d SNS junctions, the Bohr-Sommerfeld quantization rule reads:

$$\pm\left(2\arccos\frac{\varepsilon}{\Delta} + \underbrace{\frac{2\varepsilon L}{v_F}}_{\text{accumulated phase difference acquired by the particle and AR hole in N}}\right) + \varphi = 2n\pi$$

for $\begin{cases} R/L \text{ moving } e \\ L/R \text{ moving } h \end{cases}$

If $\frac{v_F}{L} \ll \Delta$, the ABS spectrum is now given by

$$\varepsilon_{n\pm} = \frac{\pi v_F}{L}\left(n + \frac{1}{2} \pm \frac{\varphi}{2\pi}\right)$$

The total energy reads

$$E = -\frac{1}{2}\int d\varepsilon\, \varepsilon\, \text{th}\frac{\varepsilon}{2T} \underbrace{\sum_{n\pm} \delta(\varepsilon - \varepsilon_{n\pm})}_{\sum_{k\pm}\frac{L}{\pi v_F} e^{-2i\pi k\left(\frac{L\varepsilon}{\pi v_F} - \frac{1}{2} \mp \frac{\varphi}{2\pi}\right)}}$$

Poisson summation formula

$$e^{-2i\pi k\left(\frac{L\varepsilon}{\pi v_F} - \frac{1}{2} \mp \frac{\varphi}{2\pi}\right)} = (-)^k e^{\pm ik\varphi} e^{-i\frac{2\varepsilon L}{v_F}k}$$

$$I = 2e\frac{\partial E}{\partial \varphi} = \sum_{k\pm} \pm i \frac{eL}{\pi v_F} k (-)^k e^{\pm ik\varphi} \underbrace{\int d\varepsilon\, \varepsilon\, \text{th}\frac{\varepsilon}{2T} e^{-i\frac{2\varepsilon L}{v_F}k}}_{+i\frac{v_F}{2L}\cdot\frac{\partial}{\partial k}\cdot\underbrace{\int d\varepsilon\, \text{th}\frac{\varepsilon}{2T} e^{-i\frac{2\varepsilon L}{v_F}k}}_{2\pi\times\frac{-iT}{\text{sh}\left(\pi T\frac{2Lk}{v_F}\right)}}}$$

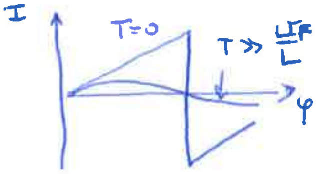
$$+i\frac{v_F}{2L}\frac{\partial}{\partial k}\left(2\pi\times\frac{-iT}{\text{sh}\left(\pi T\frac{2Lk}{v_F}\right)}\right) = -2\pi^2T^2\frac{\text{ch}\,\pi T\frac{2Lk}{v_F}}{\text{sh}^2\pi T\frac{2Lk}{v_F}}$$

$$= \frac{4eL}{\pi v_F}\sum_{k=1}^{\infty}(-)^{k+1} k \sin k\varphi\, \frac{2\pi^2T^2\,\text{ch}(\dots)}{\text{sh}^2(\dots)}$$

$$= \frac{2ev_F}{\pi}\sum_{k=1}^{\infty}\frac{(-)^{k+1}}{k}\sin k\varphi \times \left.\frac{x_k^2\,\text{ch}\,x_k}{\text{sh}^2 x_k}\right|_{x = \frac{2\pi TL}{v_F}k}$$

$T = 0$ $\quad I(\varphi) = \frac{ev_F}{\pi}\varphi \quad |\varphi| < \pi$ $\quad$ sawtooth behavior.

$T \gg \frac{v_F}{L}$ $\quad I(\varphi) \approx 16\frac{eT^2L}{v_F}e^{-\frac{2\pi TL}{v_F}}\sin\varphi$



* $\frac{v_F}{L}$ now sets the scale for minigap (i.e gap in the DOS of N metal) and critical current at $T=0$

* The critical current does not vanish at $T > \frac{v_F}{L}$ i.e. the proximity effect extends on long scale $\frac{v_F}{T}$

b_ diffusive systems

~~in the ballistic case, the wavefunction associated with the propagation of the Andreev pair reads~~

The coherence length of the Andreev pair thus extends on the scale $\xi^c_\varepsilon = \frac{v_F}{\varepsilon}$ in a ballistic system

(ε energy of incident electron and AR hole)

In diffusive systems $\xi^d_\varepsilon = \sqrt{\ell \; \xi^c_\varepsilon} = \sqrt{\frac{D}{\varepsilon}}$

ℓ = elastic mean free path

D = diffusion coefficient

2 - F/S hybrids

a - Oscillating proximity effect

- The electron and AR hole originate from a singlet Cooper pair. Thus they have opposite spins. ~~and~~
  The ABS energy is thus determined by

$$\mp \frac{2(\varepsilon \pm h) L}{v_F} + \varphi = 2n\pi$$

$h =$ Zeeman field

in long junctions.

i.e $\quad \varepsilon_{n\pm\sigma} = \frac{\pi v_F}{L}\left(n + \frac{1}{2} \pm \frac{\varphi}{2\pi} + \sigma \frac{hL}{\pi v_F}\right)$

- $I_{SFS}(\varphi) = \frac{1}{2}\left\{I_{SNS}\left(\varphi + \frac{2hL}{v_F}\right) + I_{SNS}\left(\varphi - \frac{2hL}{v_F}\right)\right\}$

at $\quad T \gg \frac{v_F}{L}$

$$I_{SFS}(\varphi) \propto \sin\varphi \cos\left(\frac{2hL}{v_F}\right)$$

$\downarrow$

oscillations on the scale $\xi_h^c = \frac{v_F}{L}$

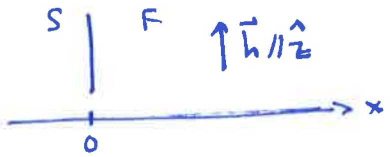
NB: in diffusive junctions, I_c both oscillates and decays on the scale $\xi_F^d = \sqrt{\frac{D}{h}}$

As $h \gg T_c$ in usual ferromagnets $\quad \xi_F \ll \xi_{T_c} < \xi_T$

i.e, the proximity effect is short ranged.

Oscillations: a π-phase difference is spontaneously established when $I_c < 0$

b - long range triplet proximity effect

• 

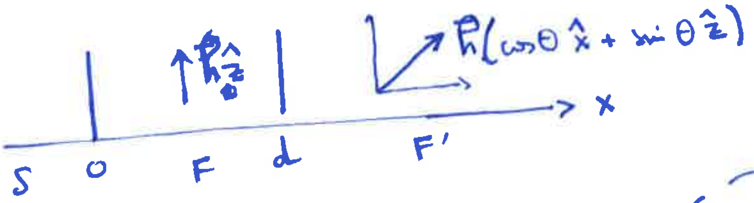
the wave function associated with the Andreev pair generated from a singlet Cooper pair is :

$$\Psi_\varepsilon(x) \sim e^{\frac{2i\varepsilon x}{v}} \left(e^{\frac{2ihx}{v}} |\uparrow\downarrow\rangle - e^{-\frac{2ihx}{v}} |\downarrow\uparrow\rangle \right)$$

$$\sim e^{\frac{2i\varepsilon x}{v}} \Big[\underbrace{\cos\frac{2hx}{v} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)}_{S=0} + \underbrace{i \sin\frac{2hx}{v} \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)}_{S=1 \quad S_z = 0} \Big]$$

In the S/F structure, only spin rotations along $\hat{z}$ axis leave the system invariant $\longrightarrow$ S_z is conserved

$\longrightarrow$ triplet correlations are induced (but still without net spin along z axis

• 

$x > d$

$$|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle = \cos\theta \left(|\nearrow\swarrow\rangle + |\swarrow\nearrow\rangle \right) + \sin\theta \left(|\nearrow\nearrow\rangle - |\swarrow\swarrow\rangle \right)$$

$$|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle = |\nearrow\swarrow\rangle - |\swarrow\nearrow\rangle$$

$\longrightarrow$ short ranged in F'

but a long range component is generated for Andreev pairs with parallel spins in F' :

$$\Psi_\varepsilon(x) \sim i\, e^{\frac{2i\varepsilon x}{v}} \sin\frac{2hd}{v} \sin\theta \left(|\nearrow\nearrow\rangle - |\swarrow\swarrow\rangle \right)$$

long range triplet proximity effect requires

- non collinearity $\quad \theta \neq 0, \text{ or } \pi$
- an intermediate F layer with optimal thickness $d \sim \xi_F$

• to observe a Josephson current, the ~~Andreev~~ pairs ~~...~~ with parallel spins should recombine into singlet pairs $\longrightarrow$ 3 layers S/F'/F/F''/S junction

IV_ Topological superconductivity

1_Majorana fermions

- Majorana found real solutions of the Dirac equation (1937)
  ↳ fermion which is its own antiparticle : $\gamma = \gamma^+$
- A Dirac fermion can always be viewed as a combination of two Majorana fermions:

$$\begin{cases} c = \frac{1}{\sqrt{2}}(\gamma_1 + i\gamma_2) \\ c^+ = \frac{1}{\sqrt{2}}(\gamma_1 - i\gamma_2) \end{cases} \longleftrightarrow \begin{cases} \gamma_1 = \frac{1}{\sqrt{2}}(c + c^+) \\ \gamma_2 = -\frac{i}{\sqrt{2}}(c - c^+) \end{cases}$$

with $\gamma_1^2 = \gamma_2^2 = \frac{1}{2}$ and $c^+c = i\gamma_1\gamma_2 + \frac{1}{2}$

- no fundamental particle is known to be a Majorana fermion: neutrino ?
- emergent low energy excitations in condensed matter physics:

  FQHE $\nu = 5/2$
  B phase of Helium 3 superfluid
  → various models involving superconductivity

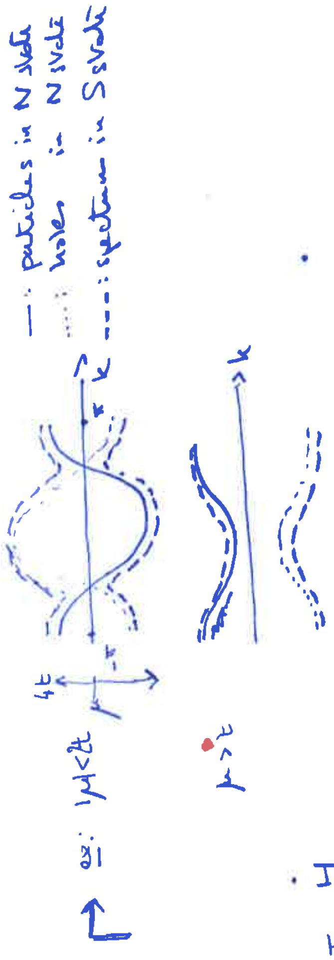
2_ The Kitaev model (2001)

- In the BCS approach of Sec I.1-, we found $\gamma_\uparrow(E) = \gamma_\downarrow^+(-E)$ in a spinless model:
  i.e. look for Majorana fermions in a spinless model :

$$H = \sum_n -t(c_n^+ c_{n+1} + c_{n+1}^+ c_n) - \mu c_n^+ c_n + \Delta(c_n^+ c_{n+1}^+ + c_{n+1} c_n)$$

1d tight binding model on N sites

t : hopping
μ : chemical potential
Δ : pair potential that cannot act on-site (Pauli principle)
→ $\Delta(k) \propto \sin k_x$ "p-wave" symmetry for the Cooper wave function to be symmetric



- Infinite chain: $H = \sum_k (c_k^+ \; c_{-k}) \begin{pmatrix} -2t\cos k - \mu & \Delta \sin k \\ \Delta \sin k & 2t\cos k + \mu \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^+ \end{pmatrix}$

  the spectrum is $E_k = \pm\sqrt{(2t\cos k + \mu)^2 + \Delta^2 \sin^2 k}$. It is always gapped except when $|\mu| = 2t$ → What happens when the gap closes and reopens (as μ varies)?

We now introduce Majorana fermions such that

$$c_n = \frac{1}{\sqrt{2}}\left(\Gamma_{An} + i\Gamma_{Bn}\right)$$
$$c_n^+ = \frac{1}{\sqrt{2}}\left(\Gamma_{An} - i\Gamma_{Bn}\right)$$

$$\hookrightarrow H = -i\sum_n\left\{(t+\Delta)\Gamma_{An}\Gamma_{Bn+1} - (t-\Delta)\Gamma_{Bn}\Gamma_{An+1}\right\} - \mu\sum_n\left(i\Gamma_{An}\Gamma_{Bn} + \frac{1}{2}\right)$$

• specific cases:

• $t = \Delta = 0$ $\quad H = -\mu \sum_i c_i^+ c_i$

$\hookrightarrow$ unique ground state (vacuum) excitations cost energy $|\mu|$) $\leftarrow$ trivial state

• $t = \Delta$ $\quad \mu = 0$ $\quad H = -it\sum_n \Gamma_{An}\Gamma_{Bn+1} = t\sum_n d_n^+ d_n$

where $d_n = \frac{1}{\sqrt{2}}\left(\Gamma_{Bn+1} + i\Gamma_{An}\right)$

$\hookrightarrow$ Majorana modes on neighbouring sites are coupled
excitations from vacuum cost energy $|t|$
but ~~these states~~ Γ_{AN} and Γ_{B1} do not appear in the Hamiltonian
$\Rightarrow$ a highly non local fermion with zero energy is formed out of these edge states.) $\leftarrow$ topologically non trivial state.

• topological transition:

$|\mu| < 2t$: 1 pair of Majorana end states (not localized on a single site in general)

$|\mu| > 2t$: • no zero energy state

NB: The zero energy state corresponds to a different parity of the total number of electrons whether it is filled or empty (up to charge $2e$ absorbed by the condensate)

NB: topological index $n \in \mathbb{Z}_2 = \{0,1\}$

n = # crossings of the Fermi surface in N state (at $k>0$)
= # pairs of end states

3. Experimental realizations

- Helical edges of a quantum spin Hall insulator (cf. L. Lévy's lecture)

$$H = \int dx \Big[-i v_F \big[\psi_R^+ \partial_x \psi_R - \psi_L^+ \partial_x \psi_L\big] - \mu\big(\psi_R^+\psi_R + \psi_L^+\psi_L\big)$$
$$+ h\big[\psi_R^+ \psi_L + \psi_R^+ \psi_L\big) + \Delta\big(\psi_R^+\psi_L^+ + \psi_L\psi_R\big)\Big]$$

R-movers with spin $\uparrow$
L-movers with spin $\downarrow$
h = transverse field
Δ = effective pairing induced by proximity with a superconductor

ex: HgTe/CdTe or InAs/GaSb heterostructures

Introducing Pauli matrices $\begin{cases} \sigma_i \text{ in } R/L \text{ space} \\ \tau_i \text{ in particle/hole space} \end{cases}$ we obtain the Bogoliubov Hamiltonian

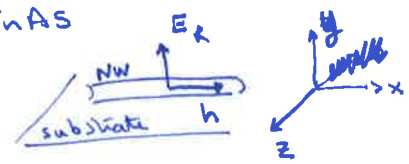
$$H = (v_F k\, \sigma_3 - \mu)\tau_z + h\sigma_x + \Delta \tau_x$$
$$\hookrightarrow H^2 = v_F^2 k^2 + \mu^2 + h^2 + \Delta^2 - 2 v_F k \mu\, \sigma_z - 2h\mu\, \sigma_x \tau_z + 2h\Delta \sigma_x \tau_x$$
$$\hookrightarrow \big[H^2 - (v_F^2 k^2 + \mu^2 + h^2 + \Delta^2)\big]^2 = 4\big[(v_F k \mu)^2 + (h\mu)^2 + (h\Delta)^2\big]$$

i.e. spectrum $\quad E_k = \pm\sqrt{v^2k^2 + \mu^2 + h^2 + \Delta^2 (\pm)' 2\sqrt{v^2k^2\mu^2 + h^2(\mu^2+\Delta^2)}}$

with a gap at $k=0$ $\quad 2\left| h - \sqrt{\mu^2+\Delta^2}\right|$

$\to$ trivial insulator with a "spin" gap if $h > \sqrt{\mu^2+\Delta^2}$
$\to$ topological superconductor if $h < \sqrt{\mu^2+\Delta^2}$
(p wave superconductor with $\Delta(k) = \Delta\, \mathrm{sgn}(k)$)

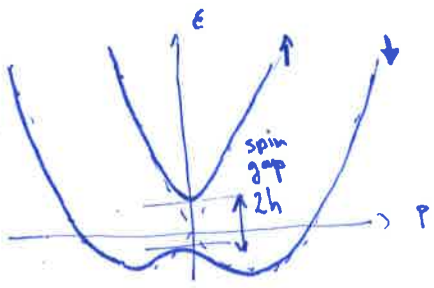
- Semiconducting nanowires ex: InSb, InAs



+

normal state : $\quad H = \dfrac{p^2}{2m} + \underbrace{\alpha p \sigma_z}_{\text{Rashba term}} + h\sigma_x \quad$ ($h /\!/ E_R$ also works)

$$= \frac{1}{2m}\big(p + m\alpha\sigma_z\big)^2 + h\sigma_x + \text{cste}$$



now with induced superconductivity, the Bogoliubov Hamiltonian reads

$$H = \frac{1}{2}\sum_p \psi_p^+ \left[\left(\frac{p^2}{2m} - \mu + \alpha p \sigma_z\right)\tau_z + \Delta \tau_x + h\sigma_x \right] \psi_p$$

with $\psi_p = \begin{pmatrix} c_{p\uparrow} \\ c_{p\downarrow} \\ c^+_{-p\downarrow} \\ -c^+_{-p\uparrow} \end{pmatrix}$

the spectrum in superconducting state again displays a gap

$$E_g = 2\left| h - \sqrt{\Delta^2+\mu^2} \right|$$

but now, $\begin{cases} h < \sqrt{\Delta^2+\mu^2} & \text{trivial superconductor} \\ h > \sqrt{\Delta^2+\mu^2} & \text{topological superconductor} \end{cases}$

<u>NB</u>: ~~[illegible]~~ effective p-wave superconductivity:

we first diagonalize the normal part of the spectrum:

$$H_N = \sum_{p\sigma} a^+_{p\sigma} \left(\frac{p^2}{2m} - \mu \pm \sqrt{h^2 + (\alpha p)^2} \right) a_{p\sigma}$$

$\sigma = \uparrow/\downarrow$ correspondent aux deux bandes de la page 22

with $\begin{pmatrix} a_{p\uparrow} \\ a_{p\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} c_{p\uparrow} \\ c_{p\downarrow} \end{pmatrix}$ avec $\cos\frac{\theta}{2} = \frac{\alpha p}{\sqrt{h^2+(\alpha p)^2}}$

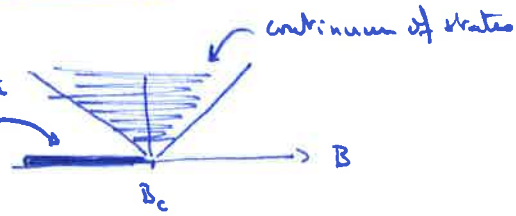
When $\alpha p_F \sim m\alpha^2, \mu \ll h$, there is effectively only one band ($\downarrow$) at the Fermi level. Projecting the Hamiltonian on that band, we obtain

$$H \approx \sum_p \left\{ \varepsilon_p a^+_{p\downarrow} a_{p\downarrow} - \underbrace{\Delta \frac{\alpha p}{h}}_{\text{p-wave pairing}} a^+_{p\downarrow} a^+_{-p\downarrow} + h.c. \right\}$$

4. Signatures of Majorana fermions

Zero-bias anomaly:

- typical spectrum



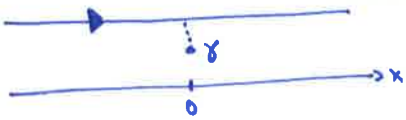
by performing a tunneling spectroscopy at the edge of a superconductor one should observe a ZBA on the topological side

- Low energy Hamiltonian (energies << superconducting gap)

$$H = \underbrace{\sum_k \epsilon_k c_k^+ c_k}_{\text{normal lead}} + \underbrace{t\,\gamma \sum_k (c_k - c_k^+)}_{\text{point contact}} \qquad (1)$$

$$\gamma = c + c^+$$

chiral 1d conductor = half infinite conductor



An incident electron at energy ε may be either reflected as an electron or as a hole

electron → no transferred current

hole → Andreev reflection process carrying a charge 2e

We rewrite in real space: (Bogoliubov de Gennes approach)

$$H = \frac{1}{2}\int dx \Big\{ \left(\psi^+(x) \;\; \psi(x)\right) \begin{pmatrix} -i v\partial_x & 0 \\ 0 & +iv\partial_x \end{pmatrix} \begin{pmatrix} \psi(x) \\ \psi^+(x) \end{pmatrix}$$

$$+ \left(c^+ \;\; c\right) \begin{pmatrix} t & -t \\ t & -t \end{pmatrix} \delta(x) \begin{pmatrix} \psi(x) \\ \psi^+(x) \end{pmatrix}$$

$$+ \left(\psi^+(x) \;\; \psi(x)\right) \begin{pmatrix} +t & +t \\ -t & -t \end{pmatrix} \delta(x) \begin{pmatrix} c \\ c^+ \end{pmatrix} \Big\}$$

This Hamiltonian can be diagonalized with the Bogoliubov transformation

$$\psi(x) = \sum_\varepsilon \left(u_\varepsilon(x)\, \gamma_\varepsilon + v_\varepsilon(x)^* \gamma_\varepsilon^+ \right)$$

$$c = \sum_\varepsilon \left(\alpha\, \gamma_\varepsilon + \beta^* \gamma_\varepsilon^+ \right)$$

provided that

$$\begin{cases} \varepsilon\, u_\varepsilon(x) = -i v_F \partial_x u_\varepsilon(x) + t\,\delta(x)(\alpha + \beta) \\ \varepsilon\, v_\varepsilon(x) = +i v_F \partial_x v_\varepsilon(x) \;\; t\,\delta(x)(\alpha + \beta) \\ \varepsilon\, \alpha = t\,[u(0) - v(0)] \\ \varepsilon\, \beta = +t\,[u(0) - v(0)] \end{cases}$$

We rewrite in real space (similar to Bogoliubov-de Gennes transformation):

$$H = \frac{1}{2}\int dx \Big\{ \left(\psi^+(x)\ \psi(-x)\right) \begin{pmatrix} -iv_F\partial_x & 0 \\ 0 & iv_F\partial_x \end{pmatrix} \begin{pmatrix} \psi(x) \\ \psi^+(-x) \end{pmatrix}$$

$$+ \left(c^+\ c\right)^{\delta(x)} \begin{pmatrix} t & -t \\ t & -t \end{pmatrix} \begin{pmatrix} \psi(x) \\ \psi^+(-x) \end{pmatrix}$$

$$+ \left(\psi^+(x)\ \psi(-x)\right)^{\delta(x)} \begin{pmatrix} t & +t \\ -t & -t \end{pmatrix} \begin{pmatrix} c \\ c^+ \end{pmatrix} \Big\}$$

We introduce the transformation

$$\psi(x) = \sum_{\varepsilon>0} u_\varepsilon(x)\,\gamma_\varepsilon + v_\varepsilon^*(-x)\,\gamma_\varepsilon^+$$

$$c = \sum_{\varepsilon>0} \alpha_\varepsilon\,\gamma_\varepsilon + \beta_\varepsilon^*\,\gamma_\varepsilon^+$$

which diagonalizes the Hamiltonian provided that

$$\begin{cases} \varepsilon u = -iv_F u' + t\,\delta(x)\,(\alpha+\beta) \\ \varepsilon v = iv_F v' - t\,\delta(x)\,(\alpha+\beta) \\ \varepsilon \alpha = t\,[u(0) - v(0)] \\ \varepsilon \beta = t\,[u(0) - v(0)] \end{cases}$$

~~Ferdinand Josephson~~

The wave function corresponding to an incident electron with energy ε, reflected as an electron with probability amplitude r and Andreev reflected as a hole with probability amplitude r_A reads

$$u_\varepsilon(x) = \begin{cases} e^{i\varepsilon x/v_F} & x<0 \\ r e^{i\varepsilon x/v_F} & x>0 \end{cases}$$

$$v_\varepsilon(x) = \begin{cases} r_A e^{-i\varepsilon x/v_F} & x<0 \\ 0 & x>0 \end{cases}$$

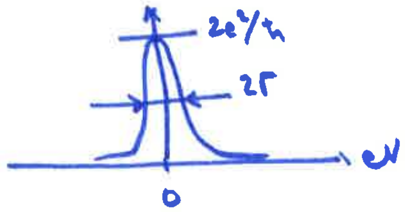
The above equations yield

$$r_A = \frac{i\Gamma}{\varepsilon + i\Gamma} \qquad \Gamma = \frac{2t^2}{v_F}$$

The Andreev current is then:

$$I = \frac{e^2}{h}\int d\varepsilon\, |r_{A\varepsilon}|^2 \left[f(\varepsilon+eV) - f(\varepsilon - eV)\right]$$

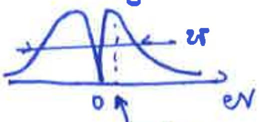
$$\frac{\partial I}{\partial V} = \frac{2e^2}{h}\,\frac{\Gamma^2}{(eV)^2+\Gamma^2} \qquad \text{at } T=0$$



quantized ZBA

NB: • many other effects may yield (not necessarily quantized) ZBA at the edge of superconductors (trivial)

• several aspects may explain non-quantized ZBA in topological superconductors.



$\varepsilon_d =$ {energy of the ABS formed with the overlap of MBS

NB: effect of the Majorana state on the other side of the nanowire!

Fractional Josephson effect

Let us couple two topological superconductors.
For simplicity, we consider two Kitaev models in the regime $t=\Delta$ and $\mu=0$, weakly coupled:

$$\bullet\,\bullet\,\bullet\,\bullet\,\bullet\,\bullet \overset{t'}{\frown} \bullet\,\bullet\,\bullet\,\bullet\,\bullet\,\bullet$$

The tunneling term between the chain is $H_T = t' e^{i\frac{\varphi}{2}} c_0^\dagger c_1 + h.c.$

where $c_0 \approx \frac{1}{\sqrt{2}}\Gamma_{A0}$ and $c_1 \approx \frac{1}{\sqrt{2}}\Gamma_{B1}$, ie $H_T \approx i t' \cos\frac{\varphi}{2}\,\Gamma_{A0}\Gamma_{B1}$

Defining $d = \frac{1}{\sqrt{2}}(\Gamma_{B1} + i\Gamma_{A0})$, $H_T = t'\cos\frac{\varphi}{2}\left(d^\dagger d - \frac{1}{2}\right)$

↳ ABS with 4π periodic energy
(energy → 0 if $t' \to 0$)

↳ 4π periodic Josephson relation if the occupation of the ABS does not change

$$I(t) = P(t)\, I_0 \sin \frac{\varphi(t)}{2}$$

$\varphi(t) = 2eV_{dc}t \longrightarrow I(t) \sim \sin(eVt)$

$\varphi(t) = 2eV_{dc} + \frac{2eV_{ac}}{\omega} \sin\omega t \longrightarrow$ Shapiro steps if $eV_{dc} = n\omega$

only even steps compared with conventional JJ (cf p.13)

NB: these effects are lost if the parity is not conserved
⟷ inelastic process due to $\begin{cases} \text{coupling to a bath of fermions \& bosons} \\ \text{nonadiabatic effects } (V(t) \neq 0) \end{cases}$

Useful textbooks on superconductivity:

- Introduction to superconductivity, M. Tinkham.
- Superconductivity of metals and alloys, P.G. De Gennes.
- Fundamentals of the theory of metals, A.A. Abrikosov.

Recent textbooks including mesoscopic superconductivity:

- Theory of fluctuations in superconductors, A. Larkin and A. Varlamov.
- Quantum transport, Yu. Nazarov and Ya. Blanter.
- The physics of nanoelectronics, T. T. Heikilla.

Pedagogical reviews on some technical aspects

- Quasiclassical Green's function approach to mesoscopic superconductivity, W. Belzig et al., Superlattices and Microstructures, **25**, 1251 (1999).
- Scattering Theory of Mesoscopic Superconductivity, S. Datta, P. F. Bagwell and M. P. Anantram, Physics of Low Dimensional Structures, **3**, 1 (1996).